Calculus na powierzchni

Σ - powierzchnia Riemanna, p

$p \in \Sigma$

Def Niech $\varphi: U \to \mathbb{C}$ będzie wsp. na otoczeniu p.

- Niech $\gamma_1: (-\epsilon, \epsilon) \to \Sigma$, $\gamma_2: (-\epsilon', \epsilon') \to \Sigma$ będą C^∞, $\gamma_1(0) = \gamma_2(0) = p$.

  $\gamma_1 \sim_p \gamma_2 \iff (\varphi \circ \gamma_1)'(0) = (\varphi \circ \gamma_2)'(0)$

- Niech $f_1: U_1 \to \mathbb{R}$, $f_2: U_2 \to \mathbb{R}$, C^∞, $p \in U_1 \cap U_2$.

  $f_1 \sim_p f_2 \iff D(f_1 \circ \varphi^{-1})_p = D(f_2 \circ \varphi^{-1})_p$

Fakt $\sim$ nie zależy od wyboru φ

Def $T_p\Sigma = \{[\gamma]_{\sim_p}\}$; $T_p^*\Sigma = \{[f]_{\sim_p}\}$.

Dzn: $[f]_{\sim_p} =: [df]_p \in T_p^*\Sigma$.

$\boxed{T_p^*\Sigma}$ $T_p^*\Sigma$ jest p. liniową (bo funkcje f można dodawać i mnożyć przez skalary)

$z = x + iy$ - wsp. na ot. p

$f(z) \rightsquigarrow f(x,y) \rightsquigarrow [df]_p = \frac{\partial f}{\partial x}\Big|_p [dx]_p + \frac{\partial f}{\partial y}\Big|_p [dy]_p$ $\qquad f \sim f_x(p) \cdot x + f_y(p) \cdot y$

$[dx]_p, [dy]_p$ - baza $T_p^*\Sigma$

$\boxed{T_p\Sigma}$

$$T_p\Sigma \ni [\gamma]_p \longmapsto (T_p^*\Sigma \to \mathbb{R}) = (T_p^*\Sigma)^*$$
$$[df]_p \longmapsto (f \circ \gamma)'(0)$$

$T_p\Sigma \simeq (T_p^*\Sigma)^* \rightsquigarrow$ struktura liniowa na $T_p\Sigma$

ZAD
w mapie:
$[\gamma]_p + [\eta]_p = [\varphi^{-1}(\varphi \circ \gamma + \varphi \circ \eta)]_p$

• Wiązka kostyczna: styczna

$$T^*\Sigma := \bigcup_{p\in\Sigma} T_p^*\Sigma \qquad T\Sigma = \bigcup_p T_p\Sigma$$

• gładkie cięcie $T^*\Sigma$, czyli 1-forma: $\alpha: \Sigma \to T^*\Sigma$ tże $\alpha(p) \in T_p^*\Sigma$

lokalnie: $\alpha(x,y) = \alpha_1(x,y)\,dx + \alpha_2(x,y)\,dy$

gładkość: $\alpha_1, \alpha_2 \in C^\infty$ ozn: $\alpha \in \Omega^1_\Sigma$

<u>Pd</u> $f \in C^\infty \to df$: $df(p) = [df]_p$

lokalnie: $df = f_x \cdot dx + f_y\, dy$

• Cofnięcie form: $F: \Sigma_1 \to \Sigma_2$,

$F: p \mapsto q$

$DF: T_p\Sigma_1 \to T_p\Sigma_2, \quad [\gamma]_p \mapsto [F\circ\gamma]_q$

$DF^*: T_p^*\Sigma \leftarrow T_q^*\Sigma_2, \quad [f]_q \mapsto [f\circ F]_p$

$\alpha \in \Omega^1_{\Sigma_2} \rightsquigarrow F^*\alpha \in \Omega^1_{\Sigma_1} \qquad (F^*\alpha)(p) = DF^*(\alpha(F(p)))$

• $d: \Omega^0_\Sigma \to \Omega^1_\Sigma$, $f \mapsto df$

$\to d(fg) = f\,dg + g\,df$

$\to d(\underset{f\circ F}{\underset{\parallel}{F^*f}}) = F^*(df)$

• Całka: $\gamma: [0,1] \to \Sigma$, obraz w mapie $\varphi: U \to \mathbb{C}$, $\alpha \in \Omega^1_\Sigma$

$$\int_\gamma \alpha = \int_0^1 \Big(\alpha_1(\gamma(t))\frac{d\gamma_1}{dt} + \alpha_2(\gamma(t))\cdot\frac{d\gamma_2}{dt}\Big)dt$$

$$\| \quad \langle \alpha(\gamma(t)), [\gamma]_{\gamma(t)} \rangle$$

nie zależy od mapy

2-formy

§3

- $\Lambda^2 E^* := \{ B : E \times E \to \mathbb{R} \mid \overset{\text{2-liniowa,}}{B(e,f) = -B(f,e)} \}$

$\dim E = 2 \Rightarrow \dim \Lambda^2 E^* = 1$

$\alpha, \beta \in E^* \rightsquigarrow \alpha \wedge \beta \in \Lambda^2 E^* : \quad (\alpha \wedge \beta)(e,f) = \alpha(e)\beta(f) - \alpha(f)\beta(e)$

- Weźmy $E = T_p\Sigma$, $E^* = T_p^*\Sigma$, $\Lambda^2 E^* = \Lambda^2 T_p^* \Sigma$

we wsp: $dx \wedge dy$ to baza/niezerowy element $\Lambda^2 T_p^* \Sigma$

- gładka 2-forma to $\omega : \Sigma \to \Lambda^2 T^* \Sigma = \bigcup_p \Lambda^2 T_p^* \Sigma$, $\omega(p) \in \Lambda^2 T_p^*$

Lokalnie: $\omega = f(x,y)\, dx \wedge dy = f(x,y)\, dx\, dy \quad (f \in C^\infty)$

Zmiana wsp.:

$$f(x(x',y'), y(x',y'))\left(\frac{\partial x}{\partial x'} dx' + \frac{\partial x}{\partial y'} dy'\right) \wedge \left(\frac{\partial y}{\partial x'} dx' + \frac{\partial y}{\partial y'} dy'\right)$$

$$= \underbrace{\left(\frac{\partial x}{\partial x'} \cdot \frac{\partial y}{\partial y'} - \frac{\partial x}{\partial y'} \cdot \frac{\partial y}{\partial x'}\right)}_{\text{Jacobian } J(x,y/x',y')} dx' \wedge dy'$$

- Całka: Jeśli $\operatorname{supp} \omega \subseteq U$ mapowym, to $\int_\Sigma \omega = \int_U \omega = \int_{\mathbb{R}^2} f(x,y)\, dx\, dy$

to nie zależy od wyboru ukł. wsp.

Ogólniej: $\omega \in \Omega^2_c(\Sigma) \Rightarrow \operatorname{supp} \omega \subseteq \bigcup_i U_i$ $\boxed{(\hat{c}\omega)}$

dobieramy rozkład 1: $\chi_i \in C_c^\infty(U_i)$, $\sum \chi_i \equiv 1$ na $\operatorname{supp} \omega$

$$\int_\Sigma \omega := \sum_i \int_{U_i} \chi_i \omega$$

- Def różniczka $d: \Omega^1(\Sigma) \to \Omega^2(\Sigma)$, liniowa, t.że

1) $\alpha_1 = \alpha_2$ na $U \Rightarrow d\alpha_1 = d\alpha_2$ na U
2) $ddf = 0$
3) $d(f\alpha) = df \wedge \alpha + f\, d\alpha$

- Lokalnie: $\alpha = f\,dx + g\,dy$,

$$d\alpha = df \wedge dx + dg \wedge dy = (f_x dx + f_y dy) \wedge dx + (g_x dx + g_y dy) \wedge dy$$
$$= (g_x - f_y)\, dx \wedge dy \qquad (\text{ćw: niezależność od wsp})$$

- TW (Stokes)

Σ - powierzchnia z brzegiem, $\alpha \in \Omega^1_c(\Sigma) \Rightarrow \int_\Sigma d\alpha = \int_{\partial\Sigma} \alpha$

D-d ćw: rozkład 1 + liniowość + rachunek w mapie. $\square$

- ZAd: $f, g \in C^\infty \mathbb{R}^2$. Czy istnieje F: $F_x = f$, $F_y = g$?

i.e. czy istnieje F: $dF = f\,dx + g\,dy =: \alpha$

TAK $\Leftrightarrow d\alpha = 0$, i.e. $f_y = g_x$

$\Rightarrow ddF = 0$

$\Leftarrow$

$$F(x,y) = \int_{(0,0)}^{(x,y)} \alpha = \int_0^y g(0,t)\,dt + \int_0^x f(t,y)\,dt \rightsquigarrow F_x = f$$

$$F(x,y) = \int_{\infty}^{(x,y)} \alpha \dashrightarrow F_y = g$$

$$\int_\Gamma \alpha - \int_\Gamma \alpha = \int_{\square} \alpha = \int_{\blacksquare} d\alpha = 0.$$

Kohomologie deRhama:

$$\Omega^0(\Sigma) \xrightarrow{d^0} \Omega^1(\Sigma) \xrightarrow{d^1} \Omega^2(\Sigma) \qquad dd=0$$

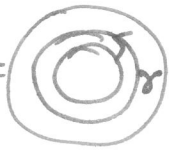
$$H^0(\Sigma) = \ker d^0$$

$$H^1(\Sigma) = \ker d^1 / \operatorname{im} d^0 \quad \leftarrow$$

$$H^2(\Sigma) = \Omega^2(\Sigma) / \operatorname{im} d^1$$

Fakt: $H^1(\mathbb{R}^2)=0 \qquad H^0(\Sigma) = \mathbb{R}^{\#\text{skł. spójnych } \Sigma}$

Pd.

1) $A =$ $\gamma =$ 

$d\theta \in \Omega^1(A)$

$dd\theta = 0$

$\int_\gamma d\theta = \theta(\gamma(1)) - \theta(\gamma(0)) = 2\pi \neq 0$

$\Rightarrow d\theta \neq dP$, bo inaczej $\int_\gamma d\theta = \underset{\overset{\shortparallel}{0}}{P(\gamma(1)) - P(\gamma(0))}$

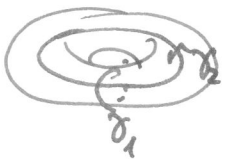
2) $H^1(S^2)$:

$\alpha \in \Omega^1(S^2)$, $d\alpha = 0$

$\alpha|_U = df_U$, $\alpha|_V = df_V$ $(U \simeq V \simeq \mathbb{R}^2)$

na $U \cap V$: $d(f_U - f_V) = 0$, $f_U - f_V = \text{const} = c$

$f := f_U \cup (c + f_V)$, $df = \alpha \Rightarrow H^1(S^2) = 0$

3) $T^2 = \underset{\theta}{S^1} \times \underset{\varphi}{S^1}$ 

$H^1(T) \ni [\alpha] \longmapsto \left(\int_{\gamma_1} \alpha, \int_{\gamma_2} \alpha\right) \in \mathbb{R}^2$ jest izo!

epi: $d\theta \mapsto (1,0)$, $d\varphi \mapsto (0,1)$

mono: $d\alpha = 0$, $\int_{\gamma_1} \alpha = \int_{\gamma_2} \alpha = 0$, $\alpha = P\,d\theta + Q\,d\varphi$

$\int_{\gamma_1} \alpha = 0 \Rightarrow \int_0^{2\pi} P(\theta,\varphi)\,d\theta = 0$; $f(\theta,\varphi) = \int_0^\theta P(\xi,\varphi)\,d\xi$, $f_\theta = P$

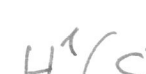
$\bar\alpha = \alpha - df = \bar Q\, d\varphi$

$d\bar\alpha = d\alpha - d\,df = 0 \Rightarrow \bar Q$ nie zależy od θ, $\bar Q = \bar Q(\varphi)$

$$0 = \int_{\gamma_2} \bar\alpha = \int_0^{2\pi} \bar Q(\theta,\xi)\,d\xi \rightsquigarrow g(\theta,\varphi) := \int_0^{\varphi} \bar Q(\theta,\xi)\,d\xi$$

$$dg = \bar Q\, d\varphi, \quad \alpha = df + dg$$

$H^1(\mathbb{R}\setminus\{p_1,\dots,p_d\})$

ćw: $H^1(S^1\times\mathbb{R})$, $H^1($  $)$, $H^1($$)$ ... $\Big|\leftarrow$

$H^*_c(\Sigma)$ – Ω^*_c zamiast Ω^*

deg!

$\int_\Sigma : H^2_c(\Sigma) \to \mathbb{R}$ (dobrze określone, bo $\int_\Sigma d\alpha = 0$)

<u>Fakt</u> $\int_\Sigma : H^2_c(\Sigma) \to \mathbb{R}$ jest izo (Σ spójna, zorientowana)

1) $\Sigma = \mathbb{R}^2$ Niech $\omega \in \Omega^2_c(\mathbb{R}^2)$, $\int_{\mathbb{R}^2}\omega = 0$, $\omega = R(x,y)\,dx\,dy$

$r(x) := \int_{-\infty}^{\infty} R(x,t)\,dt$, $\varphi =$ $\int\varphi = 1$

$\bar R(x,y) = R(x,y) - r(x)\varphi(y)$; $\int_{-\infty}^{\infty} \bar R(x,y)\,dx = 0$

$P(x,y) := \int_{-\infty}^{y} \bar R(x,t)\,dt$, $P \in C^\infty_c\mathbb{R}^2$, $P_y = \bar R$

$Q(x,y) := \varphi(y)\int_{-\infty}^{x} r(t)\,dt$, $Q \in C^\infty_c\mathbb{R}^2$, $Q_x = \varphi(y)r(x)$

$R(x,y) = \bar R(x,y) + r(x)\varphi(y) = P_y + Q_x$

$\omega = d(-P\,dx + Q\,dy)$

2) $\operatorname{supp}\omega \subseteq K \subseteq \underbrace{U_1}_{U} \cup \dots \cup \underbrace{U_n}_{V}$; indukcja ż. n.

$(\int\omega = 0)$ spójny zwarty

bzo: $U \cap V \cap K \neq \emptyset$

$f \in C^\infty_c(U)$, $g \in C^\infty_c(V)$, $f + g \equiv 1$ na K

$\omega = f\omega + g\omega$, $I := \int_\Sigma f\omega = -\int_\Sigma g\omega$

weźmy $\tau \in \Omega^2_c(U \cap V)$, $\int_\Sigma \tau = 1$

$$\underset{d\alpha}{\underset{\shortparallel}{f\omega - I\tau}} \text{ w } U \qquad \underset{d\beta}{\underset{\shortparallel}{g\omega + I\tau}} \text{ w } V \qquad \omega = d(\alpha+\beta)$$

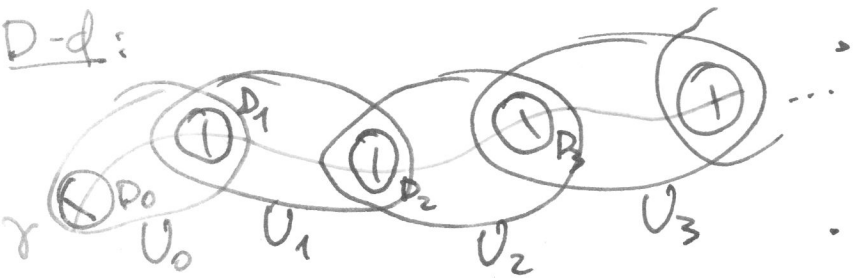
$\square$

γ - pętla w $\Sigma \Rightarrow I_\gamma : H^1(\Sigma) \to \mathbb{R}$, $\alpha \mapsto \int_\gamma \alpha$

$\theta \in \Omega^1_c(\Sigma)$, $d\theta = 0 \Rightarrow J_\theta : H^1(\Sigma) \to \mathbb{R}$, $\alpha \mapsto \int_\Sigma \theta \wedge \alpha$

<u>TW</u> $\forall \gamma \exists \theta \quad I_\gamma = J_\theta$

<u>D-d</u>:



- dobieramy $\omega_i \in \Omega^2_c(D_i)$, $\int \omega_i = 1$
- $\omega_{i+1} - \omega_i = d\theta_i$, $\theta_i \in \Omega^1_c(U_i)$

- $\theta = \sum \theta_i$, $d\theta = \sum d\theta_i = \sum \omega_{i+1} - \omega_i = 0$.

Pokażemy, że $J_\theta = I_\gamma$

- $\varphi \in \Omega^1(\Sigma)$, $d\varphi = 0$; bzo: $\varphi|_{D_i} = 0$

$\varphi|_{U_i} = df_i$

$$\int_\Sigma \theta_i \wedge \varphi = \int \theta_i \wedge df_i = \int -\underset{0}{\underset{\shortparallel}{d(\theta_i \wedge f_i)}} + \int d\theta_i f_i = \int \underset{\text{const na supp } \omega_i}{\underset{\shortparallel}{f_i}} (\omega_{i+1} - \omega_i) =$$

$$= f_i(\gamma(t_{i+1})) - f_i(\gamma(t_i)) = \int_{\gamma|_{[t_i, t_{i+1}]}} df_i = \int_{\gamma|_{[t_i, t_{i+1}]}} \varphi \quad ; \text{ sumujemy po } i$$

$\square$

$\theta' = \theta + df \Rightarrow J_\theta = J_{\theta'}$ $\left(\int df \wedge \alpha \underset{d\alpha = 0}{=} \int d(f \wedge \alpha) = 0 \right)$

$\boxed{\text{ZAD}}$ $[\alpha] \neq 0$ w $H^1(\Sigma) \Rightarrow \exists \gamma: \int_\gamma \alpha \neq 0.$

Wn

Σ - zwarta, zaukiista

$$\overset{H^1_c(\Sigma)}{\overset{\shortparallel}{H^1(\Sigma)}} \times H^1(\Sigma)$$

$$([\theta], [\alpha]) \longmapsto J_\theta([\alpha]) = \int_\Sigma \theta \wedge \alpha \in \mathbb{R}$$

jest 2-liniowa, antysymetryczna, nierdegenerowana

$\Rightarrow 2 \mid \dim H^1(\Sigma).$

Struktura zespolona:

$T_p^*\Sigma = \mathrm{Hom}_{\mathbb{R}}(T_p\Sigma, \mathbb{R})$

$T_p^*\Sigma^{\mathbb{C}} =: \mathrm{Hom}_{\mathbb{R}}(T_p\Sigma, \mathbb{C})$

$f: \Sigma \to \mathbb{C}$ gładka $\rightsquigarrow d_pf \in T_p^*\Sigma^{\mathbb{C}}$

Struktura zespolona na rzeczywistej p.l. V: $J: V \to V$, tże $J^2 = -\mathrm{Id}$.

Na $T_p\Sigma$ jest takie J, spełniające: f holomorficzna w $p \Rightarrow df$ $\mathbb{C}$-liniowe (ćw)

$$J[\gamma]_p = [\gamma + i(\gamma - p)]$$

Def. $\mathbb{R}$-liniowe $A: V \to \mathbb{C}$ jest $\mathbb{C}$-liniowe: $A(Jv) = iA(v)$

$\mathbb{C}$-antyliniowe: $A(Jv) = -iA(v)$

Fakt $\mathbb{R}$-liniowe $A: V \to \mathbb{C}$ zapisuje się jednoznacznie jako suma $\mathbb{C}$-liniowego i $\mathbb{C}$-antyliniowego:

$$A'(v) = \frac{1}{2}(A(v) - iA(Jv)); \quad A''(v) = \frac{1}{2}(A(v) + iA(Jv))$$

$$T_p^*\Sigma^{\mathbb{C}} = \underset{\substack{\rotatebox{90}{$\in$} \\ df}}{T_p^*\Sigma'} \oplus \underset{\substack{\rotatebox{90}{$\in$} \\ \overline{df}}}{T_p^*\Sigma''} \rightsquigarrow \Omega^1 = \Omega^{1,0} \oplus \Omega^{0,1}$$

f holo $\Rightarrow$