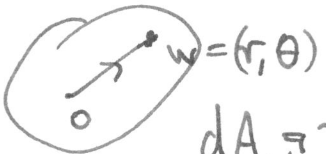
Lemat

FT1

Ω ograniczony, wypukły, otwarty $\subseteq \mathbb{R}^2$, diam $= d$, pole $= A$. ψ - gładka na otoczeniu $\bar{\Omega}$, $\bar{\psi} = \frac{1}{A}\int_\Omega \psi(x,y)\,dx\,dy$. Wtedy, dla $z \in \Omega$:

$$|\psi(z) - \bar{\psi}| \leq \frac{d^2}{2A} \int_\Omega \frac{|\nabla\psi(w)|}{|z-w|}\, dA_w$$

D-d bzo: $z = 0$, $\psi(0) = 0$.



$$\psi(w) = \int_0^r \psi_\rho \, d\rho$$

$$dA_w = r\,dr\,d\theta$$

$$\bar{\psi} = \frac{1}{A}\int_0^{2\pi}\int_0^{R(\theta)} \psi(r,\theta)\, r\,dr\,d\theta = \frac{1}{A}\int_0^{2\pi}\int_0^{R(\theta)} \underbrace{\left(\int_\rho^{R(\theta)} r\,dr\right)}_{=\frac{1}{2}(R(\theta)^2-\rho^2) \,\leq\, d^2/2} \psi_\rho \, d\rho\, d\theta$$

$$\leq \frac{d^2}{2A}\int_0^{2\pi}\int_0^{R(\theta)} \frac{\psi_\rho}{\rho}\,\rho\, d\rho\, d\theta$$

$\square$

Lemat

Założenia j.w. $\int_\Omega |\psi(z) - \bar{\psi}|^2\, dA_z \leq \left(\frac{d^3\pi}{A}\right)^2 \int_\Omega |\nabla\psi(w)|^2\, dA_w$

D-d:

Splot (na $\mathbb{R}^2$): $f*g\,(x) = \int f(y) g(x-y)\, dA_y$ "kombinacja przesunięć g"

Norma L^2: $\|g\|^2_{L^2} = \int |g(x)|^2 dA_x = \langle g, g\rangle_{L^2}$, $\langle f, g\rangle = \int_{\mathbb{R}^2} f(x) g(x)\, dA_x$

$$\|f+g\|_{L^2} \leq \|f\|_{L^2} + \|g\|_{L^2}$$

$$\|g(x-y_1) + g(x-y_2) + \ldots + g(x-y_k)\|_{L^2} \leq \sum_i \|g(\cdot - y_i)\|_{L^2} = k \cdot \|g\|_{L^2}$$

$$\left\|\sum \alpha_i g(x-y_i)\right\|_{L^2} \leq \sum |\alpha_i|\, \|g(\cdot - y_i)\|_{L^2} = \left(\sum|\alpha_i|\right)\|g\|_{L^2}$$

$$\left\|\int f(y) g(x-y)\, dA_y\right\|_{L^2} \leq \left(\int |f(y)|\, dA_y\right)\|g\|_{L^2} = \|f\|_{L^1}\|g\|_{L^2}$$

$$\|f*g\|_{L^2} \leq \|f\|_{L^1}\|g\|_{L^2}$$

$$\|f*g\|_{L^2} = \sup_{\|h\|_{L^2}\leq 1} \langle f*g, h\rangle \leq \|f\|_{L^1}\|g\|_{L^2}$$

$$|\langle f*g,h\rangle| = \left|\int\int f(y)g(x-y)h(x)\,dy\,dx\right| \leq \int|f(y)|\int|g(x-y)||h(x)|\,dx\,dy$$

$$\leq \int|f(y)|\,dy \cdot \sup_y \int|g(x-y)||h(x)|\,dx$$

$$\leq \|f\|_{L^1}\cdot\|g\|_{L^2}\|h\|_{L^2}$$

$$\langle f*g,h\rangle = \int f(y)g(\underbrace{x-y}_{u})h(\underset{\substack{\uparrow\\ u+y}}{x})\,dx\,dy = \int\int f(y)h(u+y)\,dy\,g(u)\,du$$

$$= \int f(-y)h(u-y)\,dy\,g(u)\,du = \langle g, \check{f}*h\rangle \quad \Big|\ \check{f}(x) = f(-x)$$

FT2

$$f*g = g*f: \int f(y)g(x-y)\,dA_y = \int f(x-u)\,g(u)\,dA_u$$

$u = x-y$

$y = x-u$

$$\int_\Omega \frac{|\nabla\psi(w)|}{|z-w|}\,dA_w = K*g(z)$$

$$K(w) := \begin{cases} \frac{1}{|w|}, & |w|<d \\ 0 & |w|\geq d \end{cases}, \quad g(z) = \begin{cases} |\nabla\psi(z)| & z\in\Omega \\ 0 & z\notin\Omega \end{cases}$$

Lemat ↑

$$|\psi(z)-\bar\psi| \leq \frac{d^2}{2A}|K*g(z)|$$

$$\int|\psi(z)-\bar\psi|^2\,dA_z \leq \left(\frac{d^2}{2A}\right)^2\int|K*g(z)|^2\,dA_z$$

$$\leq \frac{d^4}{4A^2}\|K\|^2_{L^1}\,\underbrace{\|g\|^2_{L^2}}_{\int|\nabla\psi(w)|^2 dA_w}$$

$$\|K\|_{L^1} = \int_{B(0,d)}\frac{1}{|x|}\,dA_x = \int_0^d\int_0^{2\pi}\frac{1}{r}\,r\,dr\,d\varphi = 2\pi d$$

$$\frac{d^4}{4A^2}\|K\|^2_{L^1} = \frac{d^4}{4A^2}\cdot 4\pi^2 d^2 = \frac{d^6\pi^2}{A^2}$$

□

D: <u>Lemat P</u> $\rho \rightsquigarrow \tilde\rho\,dx\,dy$ myślimy $\tilde\rho$: funkcja o całce 0

1) $\rho\in\Omega^2_c(U)$; $\operatorname{supp}\rho\subseteq\Omega$ j.w., $\psi\subset C^\infty\Sigma$ też ograniczony do Ω.

$$\hat\rho(\psi) = \int\psi\rho = \int(\psi-\bar\psi)\rho$$

$$|\hat\rho(\psi)| = \left|\int(\psi-\bar\psi)\rho\right| \leq \|\tilde\rho\|_{L^2(\Omega)}\|\psi-\bar\psi\|_{L^2(\Omega)}$$

$$\leq C\|\nabla\psi\|_{L^2(\Omega)}$$

$$\leq C\|d\psi\|_D$$

$\nabla\psi = (\psi_x,\psi_y)$

$d\psi = \psi_x dx + \psi_y dy$

ZAD $z = x+iy$

$|\nabla\psi|^2 = |d\psi|^2$

2) $\rho \in \Omega^2(\Sigma)$ dowolne z całką 0: $\rho = d\theta$, $\theta \in \Omega^1(\Sigma)$ FT3

Połużcie U_i, $\rho_i = d(\chi_i \theta)$ $\quad \sum \rho_i = \sum d(\chi_i \theta) = d(\sum \chi_i \theta) = d\theta = \rho$

$\operatorname{supp} \rho_i \subseteq U_i$, $\int_{U_i} \rho_i = \int d(\chi_i \theta) = 0$

$\hat{\rho}_i : C^\infty \Sigma \to \mathbb{R}$ ograniczone z 1)

$\hat{\rho} = \sum \hat{\rho}_i$ też. $\square$ (P)

Klasyka:

$K(z) = \frac{1}{2\pi} \log |z|$ – potencjał Newtona

Jeśli $f \in C_c^\infty(\mathbb{C})$, to $\Delta(K * f) = f$

Weyl (1)

Załóżmy, że $\hat{\rho}(\psi) = \langle \psi, f \rangle_D$, $f \in H$

$f = \lim \varphi_i$, $\varphi_i \in C^\infty \Sigma / \mathbb{R}$

wybieramy Ω ~~w mapowym zbiorze~~; niech ~~$\varphi_i(y) = 0$~~. $\int_\Omega \varphi_i = 0$.

wtedy $\|\varphi_i - \varphi_j\|_{L^2(\Omega)} \leq C \|\varphi_i - \varphi_j\|_D$ (nierówność P.)

Zatem (φ_i) jest ciągiem Cauchy'ego w $L^2(\Omega)$;

zbiega w $L^2(\Omega)$.

Claim $\varphi_i \longrightarrow$ w $L^2(\Sigma)$ (tzn. każdy $x \in \Sigma$ ma otoczenie Ω' t.że $\varphi_i|_{\Omega'}$ zbiegają w $L^2(\Omega')$)

D-d:

$A := \{ x \in \Sigma \mid \exists$ zbiór mapowy Ω' wokół x na którym φ_i zbiegają w $L^2 \}$

A jest otwarty, $\neq \emptyset$. Σ - spójna

Albo $A = \Sigma$, albo $\exists\ y \in \bar{A} \setminus A$.

Ω' : $\varphi_i - c_i \xrightarrow[L^2(\Omega')]{}$

A

$\varphi_i \xrightarrow[L^2(\Omega'')]{}$

w Ω'': $\varphi_i \rightarrow$, $\varphi_i - c_i \longrightarrow$ w $L^2(\Omega'')$

$c_i \to c$, $\Leftarrow$

$\varphi_i \to$ w $L^2(\Omega')$, ↯.

Zatem: $\varphi_i \xrightarrow[L^2_{loc}]{} \varphi$, ↯, $\varphi \in L^2_{loc}(\Sigma)$.

φ jest słabym rozwiązaniem $\Delta\varphi = \rho$, w następującym sensie

$$\int \psi \rho = \hat{\rho}(\psi) = \langle \psi, \underset{\lim \varphi_i}{\underset{\parallel}{f}} \rangle_D = \lim \langle \psi, \varphi_i \rangle_D = \lim \int \Delta\psi \cdot \varphi_i = \int \varphi \Delta \psi$$

gładkość jest własnością lokalną.

Lemat

$\Omega \subseteq \mathbb{C}$ ograniczony, $\rho \in \Omega^2(\Omega)$, $\varphi \in L^2(\Omega)$:

$\forall \psi \in C_c^\infty(\Omega)$
$$\int_\Omega \varphi \Delta \psi = \int_\Omega \psi \rho$$

wtedy φ jest C^∞ i spełnia $\Delta\varphi = \rho$.

Najpierw redukujemy się do przypadku $\rho = 0$:

Niech $\rho' = \rho$ na mniejszym ~~otoczeniu~~ $\Omega' \subseteq \Omega$, $\rho' \in \Omega^2_c(\Omega) \subseteq \Omega^2_c(\mathbb{C})$
$\rho' = \sigma \, dx\, dy$

Klasyka:
Niech $K(z) = \frac{1}{2\pi}\log|z|$. Wtedy $\Delta(K * \sigma) = \sigma$ / $\Delta(K*\sigma) = \rho'$!

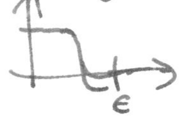
Zauważmy, że $K * \sigma \in C^\infty$

$$\int (\varphi - K*\sigma)\Delta\psi = \int \varphi\Delta\psi - \int (K*\sigma)\Delta\psi$$
$$= \int \psi\rho - \int \Delta(K*\sigma)\psi = \int\psi\rho - \int\psi\rho' = 0 \text{ dla } \psi \in C_c(\Omega')$$

Jeśli pokażemy Lemat dla $\rho = 0$, to $\varphi - K*\sigma \in C^\infty(\Omega')$, $K*\sigma \in C^\infty(\Omega')$ $\Rightarrow \varphi \in C^\infty(\Omega')$.

Załóżmy, że h jest C^∞, harmoniczne.

własność średniej: $\int_0^{2\pi} h(r,\theta)\, d\theta = 2\pi h(0)$

Dobieramy $\beta =$  tak by $2\pi \int_0^\infty r\beta(r)\, dr = 1$

Niech $B(z) = \beta(|z|)$; $B \in C_c^\infty(\mathbb{C})$, $\int_{\mathbb{C}} B(z)\, dA_z = 1$

$$\int B(-z) h(z)\, dA_z = \int_0^\infty \int_0^{2\pi} r\beta(r) h(r,\theta)\, d\theta\, dr = h(0).$$

Ogólniej: jeśli Δh ma nośnik w A, to $B*h - h = 0$ poza $N_\epsilon(A)$.

φ – nasza funkcja słabo harmoniczna (Weyl 3)

Pokażemy, że $\varphi = B*\varphi \in C^\infty$.

$$\int \psi(\varphi - B*\varphi)\,dA \qquad \Big(\text{jeśli } =0 \text{ dla } \underline{\psi \in C_c^\infty(\Omega')}, \text{ to } \varphi = B*\varphi \text{ w } \Omega'\Big)$$

$N_\epsilon(\Omega') \subseteq \Omega$

$$= \langle \psi, \varphi - B*\varphi \rangle$$

$$= \langle \psi, \varphi \rangle - \langle \psi, B*\varphi \rangle$$

$\leftarrow \langle f, g*h \rangle = \langle \check{g}*f, h \rangle$; $\check{B} = B$

$\check{g}(z) = g(-z)$

$$= \langle \psi, \varphi \rangle - \langle B*\psi, \varphi \rangle$$

$$= \langle \psi - B*\psi, \varphi \rangle$$

~~$\in$ dobieramy tak, by~~ $N_\epsilon(\operatorname{supp}\psi) \subseteq \Omega$

$\psi - B*\psi \in C_c^\infty(\Omega)$

$$= \langle \Delta(K*(\psi - B*\psi)), \varphi \rangle$$

$$= 0$$

$K*(\psi - B*\psi) = K*\psi - B*K*\psi$

$\Delta(K*\psi) = \psi \;(= 0 \text{ poza } \operatorname{supp}\psi)$

$K*\psi - B*(K*\psi) \;(= 0 \text{ poza } N_\epsilon(\operatorname{supp}\psi))$

$= K*(\psi - B*\psi) \in C_c^\infty(\Omega)$

Weyl 0

Klasyka: $K(z) = \frac{1}{2\pi} \log|z|$ potencjał Newtona

TW Jeśli $f \in C_c^\infty(\mathbb{C})$, to $\Delta(K * f) = f$

ZAD1 $\partial_x (K*f) = K * \partial_x f$, $\Delta(K*f) = K*\Delta f$

ZAD2 $U \subseteq \mathbb{C}$ podzbiór o gładkim brzegu, f lub g ma zwarty nośnik:

$$\int_U (g\Delta f - f\Delta g)\, dA = \int_{\partial U} (g D_n f - f D_n g)\, dl$$

n - jednostkowy normalny do ∂U

D-D TW

bzo: policzymy to w 0.

$$\Delta(K*f)(0) = (K*\Delta f)(0) = \int_{\mathbb{C}} K(z)\Delta f(-z)\, dA_z = \int_{\mathbb{C}} K(z)\Delta f(z)\, dA_z =$$

$$= \lim_{\epsilon \to 0} \int_{\underset{U_\epsilon}{\mathbb{C} \setminus B_\epsilon(0)}} K \Delta f\, dA = \lim_{\epsilon \to 0} \int_{U_\epsilon} (K\Delta f - f\Delta K)\, dA \underset{Z2}{=} \lim_{\epsilon \to 0} \int_{\partial U_\epsilon} (K D_n f - f D_n K)\, dl$$

$$\left[\begin{array}{l} \text{w } U_\epsilon: \Delta K = 0,\ K(z) = \frac{1}{\pi}\log|z|^2 \\ \bar\partial\partial \log|z|^2 = \bar\partial\partial \log\bar z \pm \partial\bar\partial \log z = 0 \end{array} \right]$$

$$\left| \int_{\partial U_\epsilon} K D_n f\, dl \right| \leq |\log\epsilon| \cdot C \cdot 2\pi\epsilon \xrightarrow[\epsilon \to 0]{} 0$$

$$\int_{\partial U_\epsilon} f D_n K\, dl = \frac{1}{2\pi} \int_{\partial U_\epsilon} f \frac{1}{\epsilon}\, dl \to f(0)$$

$\square$