Sub-Laplacians of holomorphic L^p-type on exponential solvable groups

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Abstract

Let L denote a right-invariant sub-Laplacian on an exponential, hence solvable Lie group G, endowed with a left-invariant Haar measure. Depending on the structure of G, and possibly also that of L, L may admit differentiable L^p-functional calculi, or may be of holomorphic L^p-type for a given $p \neq 2$. By "holomorphic L^p-type" we mean that every L^p-spectral multiplier for L is necessarily holomorphic in a complex neighborhood of some non-isolated point of the L^2-spectrum of L. This can in fact only arise if the group algebra $L^1(G)$ is non-symmetric.

Assume that $p \neq 2$. For a point ℓ in the dual $\mathfrak{g}^*$ of the Lie algebra $\mathfrak{g}$ of G, we denote by $\Omega(\ell) = Ad^*(G)\ell$ the corresponding coadjoint orbit. We prove that every sub-Laplacian on G is of holomorphic L^p-type, provided there exists a point $\ell \in \mathfrak{g}^*$ satisfying "Boidol's condition" (which, by [19] is equivalent to the non-symmetry of $L^1(G)$), such that the restriction of $\Omega(\ell)$ to the nilradical of $\mathfrak{g}$ is closed. This work improves on the results in [15] in twofold ways: On the one hand, we no longer impose any restriction on the structure of the exponential group G, and on the other hand, for the case $p > 1$, our conditions need to hold for a single coadjoint orbit only, and not for an open set of orbits, in contrast to [15].

It seems likely that the condition that the restriction of $\Omega(\ell)$ to the nilradical of $\mathfrak{g}$ is closed could be replaced by the weaker condition that the orbit $\Omega(\ell)$ itself is closed. This would then prove one implication of a conjecture made in [15], according to which there exists a sub-Laplacian of holomorphic L^1 (or, more generally, L^p)-type on G if and only if there exists a point $\ell \in \mathfrak{g}^*$ whose orbit is closed and which satisfies Boidol's condition.[1]

Introduction

A comprehensive discussion of the problem studied in this article, background information and references to further literature can be found in [15]. We shall therefore content ourselves in this introduction by recalling some notation and results from [15].

If T is a self-adjoint linear operator on a Hilbertian L^2-space $L^2(X, d\mu)$, with spectral resolution $T = \int\limits_{\mathbb{R}} \lambda dE_\lambda$, and if m is a bounded Borel function on $\mathbb{R}$, then we call m an *L^p-multiplier for* T $(1 \leq p < \infty)$, if $m(T) := \int_{\mathbb{R}} m(\lambda) dE_\lambda$ extends from $L^p \cap L^2(X, d\mu)$ to a bounded operator on $L^p(X, d\mu)$. We shall denote by $\mathcal{M}_p(T)$ the space of all L^p-multipliers for T, and by $\sigma_p(T)$ the L^p- spectrum of T. We say that T is of *holomorphic L^p-type*, if there exist some non-isolated point λ_0 in the L^2-spectrum $\sigma_2(T)$ and an open complex neighborhood $\mathcal{U}$ of λ_0 in $\mathbb{C}$, such that every $m \in \mathcal{M}_p(T) \cap C_\infty(\mathbb{R})$ extends holomorphically to $\mathcal{U}$. Here, $C_\infty(\mathbb{R})$ denotes the space of all continuous functions on $\mathbb{R}$ vanishing at infinity.

Assume in addition that there exists a linear subspace $\mathcal{D}$ of $L^2(X)$ which is T-invariant and dense in $L^p(X)$ for every $p \in [1, \infty[$, and that T coincides with the closure of its restriction to

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$\mathcal{D}$. Then, if T is of holomorphic L^p-type, the set $\mathcal{U}$ belongs to the L^p-spectrum of T, i.e.

$$\overline{\mathcal{U}} \subset \sigma_p(T). \tag{0.1}$$

In particular,

$$\sigma_2(T) \subsetneq \sigma_p(T). \tag{0.2}$$

Throughout this article, G will denote an *exponential* Lie group, i.e. the exponential mapping $\exp : \mathfrak{g} \to G$ is a diffeomorphism from the Lie algebra $\mathfrak{g}$ of G onto G. Such a group is solvable [1]. The inverse mapping to exp will be denoted by log.

We fix a *left-invariant Haar measure* dg on G. If $\pi : G \to \mathcal{U}(\mathcal{H})$ is a unitary representation of G on the Hilbert space $\mathcal{H} = \mathcal{H}_\pi$, then we denote the integrated representation of $L^1(G) = L^1(G, dg)$ again by π, i.e. $\pi(f)\xi := \int_G f(g)\pi(g)\xi \, dg$ for every $f \in L^1(G),\ \xi \in \mathcal{H}$. For $X \in \mathfrak{g}$, we denote by $d\pi(X)$ the infinitesimal generator of the one-parameter group of unitary operators $t \mapsto \pi(\exp tX)$. By X^r we denote the right- invariant vector field on G, given by

$$X^r f(g) \quad := \quad \lim_{t\to 0} \frac{1}{t}[f((\exp tX)g) - f(g)].$$

For a given function f on G, we write

$$[\lambda(g)f](x) := f(g^{-1}x), \quad g, x \in G,$$

for the *left-regular* action of G. Then λ, acting on $L^2(G)$, is a unitary representation. In particular, we have

$$X^r = -d\lambda(X) \tag{0.3}$$

and

$$\pi(X^r\varphi) = -d\pi(X)\pi(\varphi) \tag{0.4}$$

for every $X \in \mathfrak{g},\ \varphi \in \mathcal{D}(G) := C_0^\infty(G)$ and every unitary representation π of G.

In the sequel, we shall usually identify $X \in \mathfrak{g}$ with the right-invariant vector field $-X^r = d\lambda(X)$, since $d\lambda$ (as $d\pi$ for any unitary representation π) is a morphism of Lie algebras. One should notice that $d\lambda(X)$ agrees with $-X$ at the identity e of G, not with X. Then (0.4) reads simply

$$\pi(X\varphi) = d\pi(X)\pi(\varphi). \tag{0.5}$$

$d\pi$ extends from $\mathfrak{g}$ to a representation π_∞ of the universal enveloping algebra $\mathfrak{u}(\mathfrak{g})$ of $\mathfrak{g}$ on the space $C^\infty(\pi)$ of all C^∞-vectors for π. Extending the convention above, we shall often identify $A \in \mathfrak{u}(\mathfrak{g})$ with the right-invariant differential operator $\lambda_\infty(A)$ on G. Notice that $\lambda_\infty(\mathfrak{u}(\mathfrak{g}))$ consists of all right-invariant complex coefficient differential operators on G.

Choose right invariant vector fields $X_1, \ldots, X_k$ of $\mathfrak{g}$ generating $\mathfrak{g}$ as a Lie algebra, and form the so-called *sub-Laplacian*

$$L = -\sum_{j=1}^{k} X_j^2.$$

By [17], [10] L is hypoelliptic and essentially self-adjoint as an operator on $L^2(G, dg)$ with domain $\mathcal{D}(G)$. We denote its closure again by L. Since G is amenable, one has

$$\sigma_2(L) = [0, \infty[. \tag{0.6}$$

In this article, we shall give sufficient conditions for such an operator to be of holomorphic L^p-type. As has been explained in [15], a necessary condition for this to happen is the non-symmetry of the underlying group. Recall that the *modular function* Δ_G on G is defined by the equation

$$\int_G f(xg)dx = \Delta_G(g)^{-1} \int_G f(x)dx, \quad g \in G.$$

We put

$$\begin{aligned} \check{f}(g) &:= f(g^{-1}), \\ f^*(g) &:= \Delta_G^{-1}(g)\overline{f(g^{-1})}. \end{aligned}$$

Then $f \mapsto f^*$ is an isometric involution on $L^1(G)$, and for any unitary representation π of G, we have

$$\pi(f)^* = \pi(f^*) \ .$$

The group G is said to be *symmetric*, if the associated group algebra $L^1(G)$ is symmetric, i.e. if every element $f \in L^1(g)$ with $f^* = f$ has a real spectrum with respect to the involutive Banach algebra $L^1(G)$.

The exponential solvable non-symmetric Lie groups have been completely classified by Poguntke [19] (with previous contributions by Leptin, Ludwig and Boidol) in terms of a purely Lie-algebraic condition (B). Let us describe this condition, which had been first introduced by Boidol in a different context [2].

Recall that the unitary dual of G is in one to one correspondence with the space of coadjoint orbits in $\mathfrak{g}*$ via the Kirillov map, which associates with a given point $\ell \in \mathfrak{g}*$ an irreducible unitary representation π_ℓ (see Section 1).

If ℓ is an element of the dual space $\mathfrak{g}^*$ of $\mathfrak{g}$, denote by

$$\mathfrak{g}(\ell) := \ker \mathrm{ad}^*(\ell) = \{X \in \mathfrak{g} : \ell([X,Y]) = 0 \ \forall Y \in \mathfrak{g}\}$$

the *stabilizer* of ℓ under the coadjoint action ad^*. Moreover, if $\mathfrak{m}$ is any Lie algebra, denote by

$$\mathfrak{m} = \mathfrak{m}^1 \supset \mathfrak{m}^2 \supset \dots$$

the descending central series of $\mathfrak{m}$, i.e. $\mathfrak{m}^2 = [\mathfrak{m}, \mathfrak{m}]$, and $\mathfrak{m}^{k+1} = [\mathfrak{m}, \mathfrak{m}^k]$. Put

$$\mathfrak{m}^\infty = \bigcap_k \mathfrak{m}^k.$$

$\mathfrak{m}^\infty$ is the smallest ideal $\mathfrak{k}$ in $\mathfrak{m}$ such that $\mathfrak{m}/\mathfrak{k}$ is nilpotent. Put

$$\mathfrak{m}(\ell) := \mathfrak{g}(\ell) + [\mathfrak{g}, \mathfrak{g}].$$

Then we say that ℓ respectively the associated coadjoint orbit $\Omega(\ell) := \mathrm{Ad}^*(G)\ell$ satisfies *Boidol's condition* (B), if

$$\text{(B)} \qquad \ell \,|_{\mathfrak{m}(\ell)^\infty} \neq 0.$$

According to [19], the group G is non-symmetric if and only if there exists a coadjoint orbit satisfying Boidol's condition.

If Ω is a coadjoint orbit, and if $\mathfrak{n}$ is the nilradical of $\mathfrak{g}$, then

$$\Omega|_\mathfrak{n} := \{\ell|_\mathfrak{n} : \ell \in \Omega\} \subset \mathfrak{n}^*$$

will denote the restriction of Ω to $\mathfrak{n}$.

In this article, we shall prove the following extension and improvement of the main theorems in [15].

Theorem 1 *Let G be an exponential solvable Lie group, and assume that there exists a coadjoint orbit $\Omega(\ell)$ satisfying Boidol's condition, whose restriction to the nilradical $\mathfrak{n}$ is closed. Then every sub-Laplacian on G is of holomorphic L^p-type, for $1 \leq p < \infty$.*

Remarks. (a) A sub-Laplacian L on G is of holomorphic L^p-type if and only if every continuous bounded multiplier $F \in \mathcal{M}_p(L)$ extends holomorphically to an open neighborhood of a non-isolated point in $\sigma_2(L)$.

For, with F, also the function $\tilde{F}(\lambda) := e^{-\lambda}F(\lambda)$ lies in $\mathcal{M}_p(L)$, since $\tilde{F}(L) = e^{-L}F(L)$, where the heat operator e^{-L} is bounded on every $L^p(G)$. Furthermore, $\tilde{F}$ lies in $C_\infty(\mathbb{R})$.

(b) If the restriction of a coadjoint orbit to the nilradical is closed, then the orbit itself is closed (see Thm. 2.2).

(c) Under the hypotheses of the theorem, we obtain in particular that the L^2-spectrum of L is strictly contained in the L^p-spectrum of L (see (0.2)). This results has been proved independently by D. Poguntke [20].

(d) What we really use in the proof is the following property of the orbit Ω :

Ω is closed, and for every real character ν of $\mathfrak{g}$ which does not vanish on $\mathfrak{g}(\ell)$, there exists a sequence $\{\tau_n\}_n$ of real numbers such that $\lim_{n\to\infty} \Omega + \tau_n\nu = \infty$ in the orbit space.

This property is a consequence of the closedness of $\Omega|_{\mathfrak{n}}$. There are, however, many examples where the condition above is satisfied, so that the conclusion of the theorem still holds, even though the restriction of Ω to the nilradical is not closed (see e.g. Section 7). We do not know whether the condition above automatically holds whenever the orbit Ω is closed.

The article is organized as follows: In Sections 1 and 2 we recall some basic facts from the unitary representation theory of exponential Lie groups (compare [1], [14]). Moreover, we prove a kind of Riemann-Lebesgue lemma for one parameter families of coadjoint orbits whose restrictions to the nilradical are closed. In the third section, we show how the irreducible unitary representations of such a group, which are in fact induced from characters of suitable polarizing subgroups, can be realized on Euclidean L^2-spaces. This will then allow for the construction of analogous, isometric representations on certain mixed L^p-spaces. Section 4 provides some auxiliary results. In Section 4.1, we prove some results on compact operators acting on mixed L^p-spaces and their spectral properties. In particular, we prove an extension of a classical interpolation theorem by Krasnoselskii for compact operators acting on mixed L^p-spaces. Moreover, making use of well-known results on approximate units of Herz-Schur multipliers for amenable groups, we prove a result on the approximation of certain convolution operators by convolutions with continuous functions with compact support. This result will later allow us to apply a transference result by Coifman and Weiss to spectral multiplier operators $F(L)$. In Section 5 we show how, in the presence of Boidol's condition, one can construct certain analytic families $\{\pi_\ell^z\}_z$ of bounded representations acting on mixed L^p- spaces. Moreover, putting $T(z) := \pi_\ell^z(h_1)$, where h_t denotes the heat kernel associated to L at time $t > 0$, we show that $\{T(z)\}_z$ is an analytic family of compact operators on a wide range of mixed L^p-spaces, so that we can apply analytic perturbation theory. Putting together all results from the preceding sections, we complete the proof of Theorem 1 in Section 6. Finally, in Section 7 we present the example announced in Remark (d).

1 Irreducible unitary representations

Let again $G = \exp \mathfrak{g}$ denote an exponential solvable Lie group and $\mathfrak{n}$ a nilpotent ideal of $\mathfrak{g}$ containing $[\mathfrak{g}, \mathfrak{g}]$. Consider a composition sequence

$$\mathfrak{g} = \mathfrak{g}_0 \supset \mathfrak{g}_1 \supset \ldots \supset \mathfrak{g}_m = \{0\}$$

for the adjoint action of $\mathfrak{g}$, so that $\mathfrak{g}_j/\mathfrak{g}_{j+1}$ is an irreducible ad($\mathfrak{g}$)-module. Since $\mathfrak{g}$ is solvable, by Lie's theorem we have $\dim \mathfrak{g}_j/\mathfrak{g}_{j+1} \leq 2$. We may and shall assume that $\mathfrak{g}_q = \mathfrak{n}$ for some q. Choose a refinement

$$\mathfrak{g} = \mathfrak{a}_0 \supset \mathfrak{a}_1 \supset \ldots \supset \mathfrak{a}_r = \{0\}$$

of the composition sequence, which means that $\dim(\mathfrak{a}_j/\mathfrak{a}_{j+1}) = 1$, and that either $\mathfrak{a}_j = \mathfrak{g}_i$ for some i, or, if $\mathfrak{a}_j$ is not an ideal of $\mathfrak{g}$, then $\mathfrak{a}_{j-1} = \mathfrak{g}_i$ and $\mathfrak{a}_{j+1} = \mathfrak{g}_{i+1}$ for some i. We call such a sequence $\{\mathfrak{a}_j\}_j$ a *Jordan-Hölder sequence* for $\mathfrak{g}$. Each $\mathfrak{a}_j$ is a subalgebra of $\mathfrak{g}$.

Let now ℓ be an element of $\mathfrak{g}^*$. Denote by $\mathfrak{a}_j(\ell)$ the subalgebra $\mathfrak{a}_j(\ell) := \{X \in \mathfrak{a}_j : \ell([X, \mathfrak{a}_j]) = \{0\}\}$, i.e. $\mathfrak{a}_j(\ell)$ is the stabilizer of $\ell|_{\mathfrak{a}_j}$ in $\mathfrak{a}_j$.

Put

$$\mathfrak{p}(\ell) := \sum_{j=0}^{r-1} \mathfrak{a}_j(\ell).$$

Then $\mathfrak{p}(\ell)$ is a so-called *Vergne-polarization* for ℓ. In particular, it is a *polarization*, i.e. a subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ of maximal possible dimension $\frac{1}{2}(\dim \mathfrak{g} + \dim \mathfrak{g}(\ell))$ such that $\ell([\mathfrak{p}, \mathfrak{p}]) = \{0\}$. Let $P(\ell) := \exp \mathfrak{p}(\ell) \subset G$. We can define the unitary character

$$\chi_\ell(p) := e^{i\ell(\log p)}, \quad p \in P(\ell),$$

of the closed subgroup $P(\ell)$, and denote by

$$\pi_\ell = \pi_{\ell, P(\ell)} := \mathrm{ind}_{P(\ell)}^G \chi_\ell$$

the unitary representation of G induced by the character χ_ℓ of $P(\ell)$. Let us briefly recall the notion of *induced representation* [1]:

If P is any closed subgroup of G, with left-invariant Haar measure dp and modular function Δ_P, denote for $F \in C_0(G)$ by $\dot{F}$ the function on G given by

$$\dot{F}(x) := \int_P F(xp) \frac{\Delta_G}{\Delta_P}(p)\, dp, \quad x \in G,$$

where Δ_G and Δ_P denote the modular functions of G and P, respectively. We shall also write $\Delta_{G,P}$ instead of Δ_G/Δ_P. Then $\dot{F}$ lies in the space

$$\begin{aligned}\mathcal{E}(G, P) := \{f \in C(G, \mathbb{C}) : f \ &\text{has compact support modulo } P,\\ &\text{and } f(xp) = (\Delta_{G,P}(p))^{-1} f(x) \quad \forall x \in G, p \in P\}.\end{aligned}$$

In fact, one can show that $\mathcal{E}(G, P) = \{\dot{F} : F \in C_0(G)\}$. Moreover, one checks that $\dot{F} = 0$ implies $\int_G F(x)dx = 0$. From here it follows that there exists a unique positive linear functional, denoted by $\oint_{G/P} d\dot{x}$, on the space $\mathcal{E}(G, P)$, which is left-invariant under G, such that

$$\int_G F(x)\, dx = \oint_{G/P} \dot{F}(x)\, d\dot{x} = \oint_{G/P} \int f(xp) \Delta_{G/P}(p)\, dp\, d\dot{x} \tag{1.1}$$

for every $F \in C_0(G)$.

Now, given ℓ and the polarizing subgroup $P = P(\ell)$, put

$$\mathcal{E}(G,P,\ell) := \{f \in C(G,\mathbb{C}) : f \text{ has compact support modulo} P, \\ \text{and } f(xp) = \overline{\chi_\ell(p)}(\Delta_{G,P}(p))^{-1/2} f(x) \ \forall x \in G, \quad p \in P\},$$

endowed with the norm

$$||f||_2 := \left(\oint_{G/P} |f(x)|^2 \, d\dot{x} \right)^{1/2}.$$

Observe that $|f|^2 \in \mathcal{E}(G,P)$. Let $\mathcal{H}_\ell = \mathcal{H}_{\ell,P(\ell)}$ denote the completion of $\mathcal{E}(G,P,\ell)$ with respect to this norm. Then $\mathcal{H}_\ell$ becomes a Hilbert space, on which G acts by left-translations isometrically, and π_ℓ is defined on $\mathcal{H}_\ell$ by

$$[\pi_\ell(g)f](x) := f(g^{-1}x) = [\lambda(g)f](x) \quad \text{for all } f \in \mathcal{H}_\ell, \ g,x, \in G.$$

It has been shown by Bernat-Pukanszky and Vergne that the unitary representation π_ℓ is irreducible, and that π_ℓ is equivalent to $\pi_{\ell'}$, if and only if ℓ and ℓ' lie on the same coadjoint orbit, i.e. if and only if $\mathrm{Ad}^*(G)\ell = \mathrm{Ad}^*(G)\ell'$ (see [1] or [14, Theorem 8]) . Moreover, every irreducible unitary representation of G is equivalent to some π_ℓ. This shows that one has a bijection

$$K : \mathfrak{g}^*/\mathrm{Ad}^*(G) \to \hat{G}, \quad \mathrm{Ad}^*(G)\ell \mapsto [\pi_\ell],$$

called the *Kirillov-map*. Here, $[\pi_\ell]$ denotes the equivalence class of π_ℓ, and $\hat{G}$ the (unitary) dual of G, i.e. the set of all equivalence classes of unitary irreducible representations of G.

2 The topology of $\hat{G}$

Suppose again G to be exponential, and denote by $C^*(G)$ the *C^*-algebra of G*, which is, by definition, the completion of $L^1(G)$ with respect to the *C^*-norm*

$$||f||_{C^*} := \sup_{\pi \in \hat{G}} ||\pi(f)||, \quad f \in L^1(G).$$

Since G is amenable, $||f||_{C^*}$ is in fact equal to $||\lambda(f)||$, where λ denotes the left-regular representation (see ([18]).

If $\pi \in \hat{G}$, π extends uniquely to an irreducible unitary representation of $C^*(G)$, also denoted by π, and we let I_π be the kernel of π in $C^*(G)$. This two-sided ideal is by definition a so-called *primitive ideal*, and we denote by $\mathrm{Prim}(G) := \{I_\pi : \pi \in \hat{G}\}$ the set of all primitive ideals of $C^*(G)$. We endow $\mathrm{Prim}(G)$ with the *Jacobson topology.* Thus a subset C of $\mathrm{Prim}(G)$ is closed if and only if C is the *hull* $h(I)$ of an ideal, i.e. $C = h(I) := \{J \in \mathrm{Prim}(G) : J \supset I\}$. For any subset A of $\mathrm{Prim}(G)$, we denote by $\ker A := \bigcap_{J \in A} J$ the *kernel* of A, which is an ideal in $C^*(G)$.

In any C^*-algebra, a closed two-sided ideal I is always the kernel of its hull, i.e.

$$I = \bigcap_{J \in \mathrm{Prim} C^*(G), J \supset I} J; \tag{2.1}$$

see e.g. [4, 2.9.7].

Now, since exponential Lie groups are so-called type I groups, the mapping

$$\iota : \hat{G} \ni [\pi] \mapsto I_\pi \in \mathrm{Prim}(G)$$

is a bijection (see [14, §6]). In particular, $\iota \circ K : \mathfrak{g}^*/\mathrm{Ad}^*(G) \to \mathrm{Prim}(G)$ is bijective.

Even more is true: If we endow $\mathfrak{g}^*/\mathrm{Ad}^*(G)$ with the quotient topology induced by the topology of $\mathfrak{g}^*$, then

$$\iota \circ K \textit{ is a homeomorphism}$$

(see [14, §3, Theorem 1]). We introduce on $\hat{G}$ a topology by pulling back the topology of $\mathrm{Prim}(G)$ via ι.

Our proof of Theorem 1 will make use of the following results, the first of which is taken from [15].

Theorem 2.1 *Suppose G is an exponential solvable Lie group, and let $\ell \in \mathfrak{g}^*$. If the orbit $\Omega(\ell) = \mathrm{Ad}^*(G)\ell$ is closed, then $\pi_\ell(C^*(G))$ is the algebra of all compact operators on $\mathcal{H}_\ell$. In particular, $\pi_\ell(f)$ is compact for every $f \in L^1(G)$.*

The second result is a kind of "Riemann-Lebesgue Lemma". Let us call an element $\nu \in \mathfrak{g}^*$ a *character*, if $\nu([\mathfrak{g}, \mathfrak{g}]) = \{0\}$.

Theorem 2.2 *Suppose G is an exponential solvable Lie group, and let $\ell \in \mathfrak{g}^*$ with coadjoint orbit $\Omega := \Omega(\ell)$. Assume that the restriction of Ω to the nilradical $\mathfrak{n}$ of the Lie algebra $\mathfrak{g}$ is closed. Then the orbit Ω is itself closed, and for any real character ν of $\mathfrak{g}$ which does not vanish on the stabilizer $\mathfrak{g}(\ell)$ of ℓ, we have that*

$$\lim_{|\tau| \to \infty} \Omega + \tau\nu = \lim_{|\tau| \to \infty} \Omega(\ell + \tau\nu) = \infty \tag{2.2}$$

in the orbit space. In particular,

$$\lim_{|\tau| \to \infty} ||\pi_{\ell+\tau\nu}(f)|| = 0 \tag{2.3}$$

for every $f \in L^1(G)$.

Proof. Let $p := \ell|_{\mathfrak{n}}$ be the restriction of ℓ to $\mathfrak{n}$. The stabilizers $G(\ell)$ and $G(p)$ of ℓ respectively of p in G are closed connected subgroups, and we have $G(\ell) \subset G(p)$. There exists a closed subset T of G such that G is the topological product of T and $G(p)$, i.e. such that the mapping

$$T \times G(p) \to G, \quad (t, u) \to t \cdot u,$$

is a homeomorphism. In the same way let S be a closed subset of $G(p)$ such that the mapping

$$S \times G(\ell) \to G(p), \quad (s, u) \to s \cdot u,$$

is a homeomorphism (see [14]). For $g \in G$ and $m \in \mathfrak{g}^*$, let us write the action of g on m as

$$\mathrm{Ad}^*(g)m := g \cdot m.$$

Put $\mathfrak{m}(\ell) := \mathfrak{g}(\ell) + \mathfrak{n}$. It is well-known that

$$G(p) \cdot \ell = \ell + \mathfrak{m}(\ell)^\perp. \tag{2.4}$$

In fact, if $H = \exp \mathfrak{h} := G(p)$, then $\mathfrak{h} = \{X \in \mathfrak{g} : \ell([X, Y]) = 0 \quad \forall Y \in \mathfrak{n}\}$. Therefore, if $Y \in \mathfrak{m}(\ell)$ and $X \in \mathfrak{h}$, then

$$\ell(e^{\mathrm{ad}X}Y) - \ell(Y) = p([X, Y]) + \frac{1}{2}p([X, [X, Y]]) + \ldots = 0,$$

since $\mathrm{ad}^*(X)p = 0$. This implies that $H \cdot \ell \subset \ell + \mathfrak{m}(\ell)^\perp$.

Moreover, since the bilinear form $B_\ell(X,Y) := \ell([X,Y])$ is non-degenerate on $\mathfrak{g}$ modulo $\mathfrak{g}(\ell)$, we have

$$\mathfrak{g}(p)/\mathfrak{g}(\ell) \simeq (\mathfrak{n} + \mathfrak{g}(\ell))^\perp = \mathfrak{m}(\ell)^\perp,$$

hence $\dim H \cdot \ell = \dim \mathfrak{m}(\ell)^\perp$. We thus obtain (2.4).

Assume now that $\lim_{n\to\infty} \Omega + \tau_n\nu = \Omega(\ell')$, for some $\ell' \in \mathfrak{g}^*$, which means that there exists a sequence $\{m_n\}_n = \{\ell_n + \tau_n\nu\}_n$ tending to ℓ' in $\mathfrak{g}^*$, where $\ell_n \in \Omega$ and $\tau_n \in \mathbb{R}$.

We can write $\ell_n = (t_n s_n)\cdot\ell$, with $t_n \in T$ and $s_n \in S$. Since $s_n \cdot \ell = \ell + q_n$ for some $q_n \in \mathfrak{m}(\ell)^\perp$, it follows that

$$\ell_n = (t_n s_n)\cdot \ell = t_n \cdot \ell + q_n,$$

and thus

$$(\ell_n + \tau_n\nu)|_{\mathfrak{n}} = t_n \cdot \ell|_{\mathfrak{n}},$$

hence

$$\ell'|_{\mathfrak{n}} = \lim_{n\to\infty} t_n \cdot \ell|_{\mathfrak{n}}.$$

Since the restriction of Ω to $\mathfrak{n}$ is closed, we have that $\ell'|_{\mathfrak{n}} = t' \cdot p$ for some $t' \in T$, and since $\Omega|_{\mathfrak{n}}$ is homeomorphic to $G/G(p) \simeq T$, it follows that

$$\lim_{n\to\infty} t_n = t'.$$

Let us now take an element $U \in \mathfrak{g}(\ell)$ such that $\nu(U) \neq 0$. Then

$$\ell'(U) = \lim_{n\to\infty} t_n \cdot \ell(U) + \lim_{n\to\infty} \tau_n\nu(U) = t' \cdot \ell(U) + \lim_{n\to\infty} \tau_n\nu(U).$$

Hence $\lim_{n\to\infty} \tau_n = \tau'$ exists, and it follows that the sequence $\{q_n\}_n$ convergences, hence also $\lim_{n\to\infty} s_n = s'$ exists. Finally

$$\ell' = (t's')\cdot\ell + \tau'\nu \in \Omega + \tau'\nu.$$

This proves (2.2), and (2.3) is an immediate consequence of (2.2) (see [4]).

Q.E.D.

3 Representations on mixed L^p-spaces

We assume again that $\mathfrak{g} = \mathfrak{g}_0 \supset \mathfrak{g}_1 \supset \ldots \supset \mathfrak{g}_q = \mathfrak{n} \supset \ldots \supset \mathfrak{g}_m = \{0\}$ is a composition sequence passing through $\mathfrak{n}$. Let us assume that $\mathfrak{n}$ is a nilpotent ideal containing $[\mathfrak{g},\mathfrak{g}]$.

Let $\ell \in \mathfrak{g}^*$, and let $\mathfrak{p}(\ell) = \mathfrak{p}$ be the Vergne-polarization for ℓ associated to a fixed Jordan-Hölder sequence

$$\mathfrak{g} = \mathfrak{a}_0 \supset \mathfrak{a}_1 \supset \ldots \supset \mathfrak{a}_r = \{0\}$$

refining this composition sequence. Then obviously $\mathfrak{p}_0 := \mathfrak{p} \cap \mathfrak{n}$ is a Vergne-polarization for $\ell_0 := \ell|_{\mathfrak{n}}$. As in the preceding proof, let $\mathfrak{g}(\ell_0) := \{X \in \mathfrak{g} : \ell_0([X,Y]) = 0 \quad \forall Y \in \mathfrak{n}\}$ be the stabilizer of ℓ_0 in $\mathfrak{g}$. Then

$$\mathfrak{p} \subset \mathfrak{g}(\ell_0) + \mathfrak{p}_0. \tag{3.1}$$

In fact, choose k such that $\mathfrak{a}_k = \mathfrak{n}$. Then, for $j \le k$ and $X \in \mathfrak{a}_j(\ell)$, we have $\ell_0([X,Y]) = 0$ for every $Y \in \mathfrak{n}$, since $\mathfrak{n} \subset \mathfrak{a}_j$. This shows that $\mathfrak{p} \subset \mathfrak{g}(\ell_0) + \sum_{j\ge k} \mathfrak{a}_j(\ell_0) = \mathfrak{g}(\ell_0) + \mathfrak{p}_0$.

Next, for every $j \ge q$, we choose a subspace $\mathfrak{v}_j$ in $\mathfrak{g}_j$ of dimension ≤ 2, such that $\mathfrak{g}_j + \mathfrak{p}_0 = \mathfrak{v}_j \oplus (\mathfrak{g}_{j+1} + \mathfrak{p}_0)$, and define the index set J as follows:

$$J := \{j \in \{q,\ldots,m-1\} : \mathfrak{v}_j \neq \{0\}\}.$$

Write J as an ordered d-tuple

$$J = \{j_1 < \ldots < j_d\},$$

where $d := \#J$, and put $\mathfrak{w}_i := \mathfrak{v}_{j_i} \subset \mathfrak{n},\ i = 1, \ldots, d$, and $\mathfrak{w} := \mathfrak{w}_1 \oplus \ldots \oplus \mathfrak{w}_d$. We shall often identify $\mathfrak{w}$ with the direct product $\mathfrak{w}_1 \times \ldots \times \mathfrak{w}_d$.

The space $\mathfrak{w}$ then forms a complementary subspace to the polarization $\mathfrak{p}_0$ in $\mathfrak{n}$, i.e.

$$\mathfrak{n} = \mathfrak{w} \oplus \mathfrak{p}_0. \tag{3.2}$$

Let us choose a linear subspace $\mathfrak{b}$ of $\mathfrak{p}$ such that

$$\mathfrak{p} = \mathfrak{b} \oplus \mathfrak{p}_0. \tag{3.3}$$

Then $\mathfrak{b} \cap \mathfrak{n} = \{0\}$, so that we may choose a subspace $\mathfrak{h}$ of $\mathfrak{g}$ containing $\mathfrak{n}$ such that

$$\mathfrak{g} := \mathfrak{b} \oplus \mathfrak{h}.$$

Then $\mathfrak{h}$ is an ideal in $\mathfrak{g}$, and we may choose a subspace $\mathfrak{a}$ of $\mathfrak{h}$ such that

$$\mathfrak{h} = \mathfrak{a} \oplus \mathfrak{n}.$$

Then we have $\mathfrak{p} \cap \mathfrak{h} = \mathfrak{p}_0$, and, by (3.3), (3.2),

$$\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{b} \oplus \mathfrak{n} = \mathfrak{a} \oplus (\mathfrak{p} + \mathfrak{n}) = \mathfrak{a} \oplus \mathfrak{w} \oplus \mathfrak{p}. \tag{3.4}$$

Let $P := \exp \mathfrak{p}$, $P_0 := \exp \mathfrak{p}_0$ and $N := \exp \mathfrak{n}$. Then the mapping

$$\Phi = \Phi_{G,P} : \mathfrak{a} \times \mathfrak{w} \times P \to G, \quad (S, (w_1, \ldots, w_d), p) \mapsto \exp(S) \exp(w_1) \ldots \exp(w_d) p,$$

with $w_j \in \mathfrak{w}_j$, is a diffeomorphism, and

$$E = E_{G/P} : \mathfrak{a} \times \mathfrak{w} \to G, \quad (S, w) \mapsto \Phi(S, w, e)$$

provides a *section* for G/P, i.e. $\mathfrak{a} \times \mathfrak{w} \ni (S, w) \mapsto E(S, w)P$ is a diffeomorphism from $\mathfrak{a} \times \mathfrak{w}$ onto G/P.

Similarly, $\mathfrak{w} \ni w \mapsto E(0, w)P_0$ is a diffeomorphisms from $\mathfrak{w}$ onto N/P_0.

We shall later make use of the "global chart" E for G/P in order to construct a more concrete realization of the induced representation π_ℓ on a Euclidean L^2- space, which will then also allow for the construction of more general representations on mixed L^p spaces. Crucial for this construction will be the subsequent analysis of "roots" on G.

To begin with, let is construct a decomposition of $\mathfrak{p}_0$ into subspaces $\mathfrak{s}_j$ subordinate to our Jordan-Hölder sequence. To this end, choose for every $j \geq q$ a subspace $\mathfrak{r}_j$ in $\mathfrak{g}_j$ of dimension ≤ 2, such that $\mathfrak{g}_j \cap \mathfrak{p}_0 = \mathfrak{g}_{j+1} \cap \mathfrak{p}_0 \oplus \mathfrak{r}_j$, and define another index set I as follows:

$$I := \{j \in \{q, \ldots, m-1\} : \mathfrak{r}_j \neq \{0\}\}.$$

Write I again as an ordered e-tuple

$$I = \{j'_1 < \ldots < j'_e\},$$

where $e := \#I$, and put $\mathfrak{s}_i := \mathfrak{r}_{j'_i},\ i = 1, \cdots, e$. Then

$$\mathfrak{p}_0 = \mathfrak{s}_1 \oplus \ldots \oplus \mathfrak{s}_e \simeq \mathfrak{s}_1 \times \ldots \times \mathfrak{s}_e,$$

and the mapping

$$\Phi_P : \mathfrak{b} \times \mathfrak{p}_0 \to P, \quad (T, Y_1, \cdots, Y_e) \mapsto \exp(T) \exp(Y_1) \cdots \exp(Y_e) \in P$$

is a diffeomorphism which identifies the Lebesgue measure on $\mathfrak{b} \times \mathfrak{p}_0$ with the Haar measure on P.

Define also for every $j = q, \cdots, m-1$ the subspace $\mathfrak{u}_j$ of $\mathfrak{g}_j$ by $\mathfrak{u}_j := \mathfrak{r}_j + \mathfrak{v}_j$. Then $\mathfrak{u}_j$ is the direct sum

$$\mathfrak{u}_j = \mathfrak{r}_j \oplus \mathfrak{v}_j,$$

and

$$\mathfrak{g}_j = \mathfrak{u}_j \oplus \mathfrak{g}_{j+1}, \quad j = q, \ldots, m-1. \tag{3.5}$$

In particular, we have

$$\mathfrak{n} = \mathfrak{u}_q \oplus \cdots \oplus \mathfrak{u}_{m-1}.$$

According to (3.5), for $j = q, \ldots, m-1$, $X \in \mathfrak{g}$ and $U \in \mathfrak{g}_j$, we may write

$$\mathrm{ad}(X)(U) = \alpha_j(X)U + U_j, \tag{3.6}$$

where U_j is the component of $\mathrm{ad}(X)(U)$ in $\mathfrak{g}_{j+1}$, and where $\alpha_j(X)$ is an endomorphism of $\mathfrak{u}_j$. Then α_j is an irreducible representation of $\mathfrak{g}$ on $\mathfrak{u}_j$, which we shall call a *root* of $\mathfrak{g}$. Since G is exponential, the eigenvalues of $\alpha_j(X)$, considered as an endomorphism of the complexification of $\mathfrak{u}_j$, are of the form $\alpha(1+i\beta)$, where α and β are real numbers. For $X \in \mathfrak{g}$ and $j = q, \ldots, m-1$, let

$$\tau_j(X) := \mathrm{tr}\ \mathrm{ad}_{\mathfrak{g}_j/\mathfrak{g}_{j+1}}(X) = \mathrm{tr}\ \alpha_j(X),$$

where by $\mathrm{ad}_{\mathfrak{g}_j/\mathfrak{g}_{j+1}}(X)$ we denote the factorized adjoint action of X on the quotient space $\mathfrak{g}_j/\mathfrak{g}_{j+1}$.

The functionals τ_j are characters of $\mathfrak{g}$, since $\mathrm{ad}_{\mathfrak{g}_j/\mathfrak{g}_{j+1}}(X) = 0$ for every $X \in \mathfrak{n}$. Since $\mathrm{ad}(\mathfrak{p})$ acts on $\mathfrak{g}/\mathfrak{p}$, one finds that for $X \in \mathfrak{p}$ the corresponding "trace of $\mathrm{ad}(X)$ modulo $\mathfrak{p}$" is given by

$$\mathrm{tr}\ \mathrm{ad}_{\mathfrak{g}_j+\mathfrak{p}/\mathfrak{g}_{j+1}+\mathfrak{p}}(X) = \varepsilon_j \tau_j(X), \quad j = q, \ldots, m-1, \tag{3.7}$$

where

$$\varepsilon_j := \frac{\dim \mathfrak{v}_j}{\dim \mathfrak{u}_j}, \quad j = q, \ldots, m-1.$$

Observe that $\varepsilon_j \neq 0$ if and only if $\mathfrak{g}_j + \mathfrak{p}/\mathfrak{g}_{j+1} + \mathfrak{p} \simeq \mathfrak{v}_j$ is non-trivial, i.e. if and only if $j \in J = \{j_1 < \ldots < j_d\}$. For $i = 1, \ldots, d$ and $T \in \mathfrak{p}$ we shall therefore put $\lambda_i(T) := \alpha_{j_i}(T)$, so that for every $w_i \in \mathfrak{w}_i$

$$\mathrm{ad}(T)(w_i) = \lambda_i(T)w_i \ \text{ modulo } \ \mathfrak{g}_{j_i+1}, \quad i = 1, \ldots, d. \tag{3.8}$$

Then, by (3.7) and (3.8), we have

$$\mathrm{tr}\ \mathrm{ad}_{\mathfrak{g}/\mathfrak{p}}(T) = \sum_{i=1}^{d} \varepsilon_{j_i} \mathrm{tr}\, \lambda_i(T), \quad T \in \mathfrak{p}. \tag{3.9}$$

Observe now that also the mapping

$$\Psi : \mathfrak{b} \times \mathfrak{a} \times N \to G, \quad (T, S, n) \mapsto \exp(T)\exp(S)n,$$

is a diffeomorphism. And, for every $R \in \mathfrak{a}$, $w = (w_1, \ldots, w_d) \in \mathfrak{w}$ and $S \in \mathfrak{a}$, $T \in \mathfrak{b} \subset \mathfrak{p}$, $n \in \mathfrak{n}$ we have

$$\begin{aligned} (\quad & \exp(T)\exp(S)n)^{-1} E(R, w) \\ = \quad & n^{-1} \exp(-S) \exp(T)^{-1} \exp(R) \exp(T) \left(\prod_{i=1}^{d} \exp(e^{-\mathrm{ad}(T)} w_i) \right) \exp(-T). \end{aligned} \tag{3.10}$$

From (3.8) and (3.10), one can deduce that

$$(\exp(T)\exp(S)n)^{-1}E(R,w) = E(R-S,\omega(R,w,T,S,n))p(R,w,T,S,n)^{-1}, \tag{3.11}$$

where $\omega : \mathfrak{a}\times\mathfrak{w}\times\mathfrak{b}\times\mathfrak{a}\times N \to \mathfrak{w}$, $\quad p : \mathfrak{a}\times\mathfrak{w}\times\mathfrak{b}\times\mathfrak{a}\times N \to P$ are analytic mappings which depend polynomially on w and n, and where $\omega = (\omega_1,\ldots,\omega_d)$, with

$$\omega_i(R,w,T,S,n) = e^{-\lambda_i(T)}(w_i) + \tilde{\omega}_i(R,w_1,\ldots,w_{i-1},T,S,n). \tag{3.12}$$

Because of (3.3) and (3.10), we have $p(R,w,T,S,n) = \exp(T) \bmod P_0$, i.e.

$$p(R,w,T,S,n) = \exp(T)\,\nu(R,w,T,S,n), \tag{3.13}$$

with $\nu(R,w,T,S,n) \in P_0 = P\cap N$.

Putting $p := \exp(T) \in P$, we therefore obtain

$$\begin{aligned}\Delta_{G,P} \quad & (p(R,w,T,S,n)) = \Delta_{G,P}(p) = \frac{[\det e^{\operatorname{ad}_{\mathfrak{g}}(T)}]^{-1}}{[\det e^{\operatorname{ad}_{\mathfrak{p}}(T)}]^{-1}} \\ = \quad & e^{-\operatorname{tr}\operatorname{ad}_{\mathfrak{g}}(T)+\operatorname{tr}\operatorname{ad}_{\mathfrak{p}}(T)} = e^{-\operatorname{tr}\operatorname{ad}_{\mathfrak{g}/\mathfrak{p}}(T)},\end{aligned}$$

hence, by (3.9),

$$\Delta_{G,P}(p(R,w,T,S,n)) = e^{-\sum\limits_{i=1}^{d}\varepsilon_{j_i}\operatorname{tr}\lambda_i(T)}. \tag{3.14}$$

In particular, if we define the real character Δ on G by

$$\Delta(\exp(X)) := \exp\left(-\sum_{j=q}^{m-1}\varepsilon_j\tau_j(X)\right) = \exp\left(-\sum_{i=1}^{d}\varepsilon_{j_i}\tau_{j_i}(X)\right), \quad X\in\mathfrak{g},$$

then

$$\Delta_{G,P}(p) = \Delta(p) \ \text{ for every } \ p\in P. \tag{3.15}$$

Now, in oder to realize the representation on a Euclidean L^2-space, we first observe that the left-invariant linear functional $\oint_{G/P} d\dot{x}$ in (1.1) is given by

$$\oint_{G/P} f(x)\,d\dot{x} = \int_{\mathfrak{a}\times\mathfrak{w}} f\circ E(R,w)\,dR\,dw \quad \forall f\in\mathcal{E}(G,P) \tag{3.16}$$

(see [14, Theorem 2]). For $f\in\mathcal{E}(G,P)$, let us put

$$\tilde{f}(x) := \Delta(x)f(x), \quad x\in G.$$

Since, by (3.15), Δ is a character of G which extends $\Delta_{G,P}$ from P to G, we have for $x\in G$ and $p\in P$

$$\tilde{f}(xp) = \Delta(xp)f(xp) = \Delta(x)\Delta(p)\Delta_{G,P}(p)^{-1}f(x) = \tilde{f}(x).$$

Thus the mapping $\mathcal{A}^2 : f\mapsto\tilde{f}$ is a linear isomorphism from $\mathcal{E}(G,P)$ onto the space

$$\begin{aligned}\tilde{\mathcal{E}}(G,P) := \{f\in C(G,\mathbb{C}) : f \ & \text{ has compact support modulo } P, \\ & \text{and } f(xp) = f(x) \quad \forall x\in G, p\in P\}.\end{aligned}$$

Identifying functions on G/P with P-right-invariant functions on G, we thus see that $\tilde{\mathcal{E}}(G,P)\simeq C_0(G/P)$. Moreover, if we define for any continuous function f with compact support on G/P its integral by

$$\int_{G/P} f(x) d\dot{x} := \int_{\mathfrak{a}\times\mathfrak{w}} \Delta^{-1}(E(R,w)) f(E(R,w))\, dR\, dw,$$

then clearly

$$\oint_{G/P} f(x)\, d\dot{x} = \int_{G/P} \tilde{f}\, d\dot{x} \quad \text{for every } f \in \mathcal{E}(G,P). \tag{3.17}$$

Comparing with (1.1), we find in particular that

$$\int_G F(x)\, dx = \int_{G/P} \int_P F(xp)\Delta(xp)\, dp\, dx \quad \text{for every } f \in C_0(G). \tag{3.18}$$

Let us define the space

$$\tilde{\mathcal{E}}(G,P,\ell) := \{f \in C(G,\mathbb{C}) : f \text{ has compact support modulo } P, \\ \text{and } f(xp) = \overline{\chi_\ell(p)} f(x)\ \forall x \in G, \quad p \in P\},$$

endowed with the norm $||\cdot||_2$ given by

$$||f||_2^2 := \int_{G/P} |f(x)|^2\, d\dot{x} = \int_{\mathfrak{a}\times\mathfrak{w}} |\Delta^{-1/2}(E(R,w)) f(E(R,w))|^2\, dR\, dw. \tag{3.19}$$

Observe that $|f|^2 \in \tilde{\mathcal{E}}(G,P)$, if $f \in \tilde{\mathcal{E}}(G,P,\ell)$. Let $\tilde{\mathcal{H}}_\ell$ denote the completion of $\tilde{\mathcal{E}}(G,P,\ell)$ with respect to this norm. It is obvious that the mapping $\mathcal{A} : f \mapsto \Delta^{1/2} f$ is a linear isomorphism between $\mathcal{E}(G,P,\ell)$ and $\tilde{\mathcal{E}}(G,P,\ell)$, which extends to an isometric isomorphism of the Hilbert space $\mathcal{H}_\ell$ onto the Hilbert space $\tilde{\mathcal{H}}_\ell$. Moreover, $\tilde{\mathcal{H}}_\ell$ is nothing but the space

$$L^2(G/P,\ell) := \{f : G \to \mathbb{C} : f \text{ is measurable, and } f(xp) = \overline{\chi_\ell(p)} f(x) \\ \text{for a.e. } x \in G \text{ and every } p \in P, \text{ s.t. } ||f||_2 < \infty\}.$$

We may therefore intertwine the representation π_ℓ with the operator $\mathcal{A}$ in order to obtain a unitarily equivalent representation $\tilde{\pi}_\ell$ on $L^2(G/P,\ell)$, given by

$$\tilde{\pi}_\ell(g) := \mathcal{A}\pi_\ell(g)\mathcal{A}^{-1}, \quad g \in G.$$

A straight-forward computation shows that $\tilde{\pi}_\ell$ is given explicitly by

$$[\tilde{\pi}_\ell(g) f](x) = \Delta(g)^{1/2} f(g^{-1}x) \text{ for all } f \in L^2(G/P,\ell),\ g, x \in G,$$

i.e.

$$\tilde{\pi}_\ell(g) = \Delta(g)^{1/2}\lambda(g), \quad g \in G. \tag{3.20}$$

For a "multi-exponent" $\underline{p} = (p, p_1, \ldots, p_d) \in [1,\infty[^{1+d}$, let us now define the *mixed L^p-space* $L^{\underline{p}}(G/P,\ell) \simeq L^p(\mathfrak{a}, (L^{p_1}(\mathfrak{w}_1, L^{p_2}(\mathfrak{w}_2, \ldots)))$ by

$$L^{\underline{p}}(G/P,\ell) := \{f : G \to \mathbb{C} : f \text{ is measurable, and } f(xp) = \overline{\chi_\ell(p)} f(x) \\ \text{for a.e. } x \in G \text{ and every } p \in P, \text{ s.t. } ||f||_{\underline{p}} < \infty\},$$

where the mixed $L^{\underline{p}}$-norm is given by

$$||f||_{\underline{p}}$$

$$:= \left(\int_{\mathfrak{a}} \left(\int_{\mathfrak{w}_1} \cdots \left(\int_{\mathfrak{w}_{d-1}} \left(\int_{\mathfrak{w}_d} \left|(\Delta^{-\frac{1}{p}} f)(E(R, w_1, \ldots, w_d))\right|^{p_d} dw_d \right)^{\frac{p_{d-1}}{p_d}} dw_{d-1} \right)^{\frac{p_{d-2}}{p_{d-1}}} \cdots dw_1 \right)^{\frac{p}{p_1}} dR \right)^{\frac{1}{p}}.$$

The space $\tilde{\mathcal{E}}(G,P,\ell)$ is dense in $L^{\underline{p}}(G/P,\ell)$, for any $\underline{p}$.

Put

$$\gamma_i(\underline{p}) := \frac{1}{p_i}, \quad i = 1,\dots,d,$$

and define the character $\delta_{\underline{p}}$ of $\mathfrak{g}$ by

$$\delta_{\underline{p}}(X) := \sum_{i=1}^{d} \gamma_i(\underline{p})\varepsilon_{j_i}\tau_{j_i}(X), \quad \text{if } X \in \mathfrak{p},$$

$$\delta_{\underline{p}}(X) := \frac{1}{p}\sum_{j=q}^{m-1} \varepsilon_j\tau_j(X), \quad \text{if } X \in \mathfrak{a}+\mathfrak{n}.$$

Observe that $\delta_{\underline{p}}$ is well-defined, since $(\mathfrak{a}+\mathfrak{n})\cap\mathfrak{p} \subset \mathfrak{n}$, and since τ_j vanishes on $\mathfrak{n}$. The corresponding character of G is given by

$$\Delta_{\underline{p}}(\exp(X)) := e^{-\delta_{\underline{p}}(X)}, \quad X \in \mathfrak{g}.$$

Notice that for $\overline{2} := (2,\dots,2)$ we have

$$\Delta_{\overline{2}} = \Delta^{1/2} \ \text{ and } \ ||\cdot||_{\overline{2}} = ||\cdot||_2. \tag{3.21}$$

Observe also that for $R \in \mathfrak{a}, w \in \mathfrak{w}$, we have

$$\Delta(E(R,w)) = \Delta(\exp(R)).$$

Let $T \in \mathfrak{b}$ and $R \in \mathfrak{a}, w \in \mathfrak{w}$. Then, by (3.11), (3.12), we have

$$\exp(T)^{-1}E(R,w) = E(R,\{e^{-\lambda_i(T)}w_i + \tilde{\omega}_i(R,w_1,\dots,w_{i-1},T,0,e)\}_{i=1}^d)\, p(R,w,T,0,e)^{-1}.$$

From the definition of the norm $||\cdot||_{\underline{p}}$ we therefore obtain

$$||\lambda(y)f||_{\underline{p}} = \Delta_{\underline{p}}(y)^{-1}||f||_{\underline{p}} \quad \text{for every } f \in L^{\underline{p}}(G/P,\ell),\ y \in P.$$

Similarly, for $S \in \mathfrak{a}$, we have that

$$\exp S^{-1}E(R,w) = E(R-S,\{w_i + \tilde{\omega}_i(R,w_1,\dots,w_{i-1},0,e)\}_{i=1}^d)p(R,w,0,e)^{-1},$$

hence

$$||\lambda(\exp S)f||_{\underline{p}} = \Delta_{\underline{p}}(\exp(S))^{-1}||f||_{\underline{p}} \quad \text{for every } f \in L^{\underline{p}}(G/P,\ell),\ S \in \mathfrak{a}.$$

Finally, if we choose $n \in N$, then of course

$$||\lambda(n)f||_{\underline{p}} = ||f||_{\underline{p}}.$$

It is now clear that we obtain an *isometric* representation $\pi_\ell^{\underline{p}}$ of G on $L^{\underline{p}}(G/P,\ell)$ by letting

$$[\pi_\ell^{\underline{p}}(g)f](x) := \Delta_{\underline{p}}(g)f(g^{-1}x), \quad g,x \in G, f \in L^{\underline{p}}(G/P,\ell),$$

i.e.

$$\pi_\ell^{\underline{p}}(g) = \Delta_{\underline{p}}(g)\lambda(g), \quad g \in G. \tag{3.22}$$

Notice that by (3.20) and (3.21), the representation $\pi_\ell^{\overline{2}}$ is unitarily equivalent to π_ℓ, i.e.

$$\pi_\ell^{\overline{2}} \simeq \pi_\ell. \tag{3.23}$$

In the sequel, we shall work with $\pi_\ell^{\overline{2}}$ in place of π_ℓ. With a slight abuse of notation, we shall therefore denote $\pi_\ell^{\overline{2}}$ simply by π_ℓ. Observe that then for every function $f \in L^1(G)$ such that $\Delta_{\underline{p}}\Delta^{-1/2}f \in L^1(G)$, the operator $\pi_\ell^{\underline{p}}(f)$ is given by the formula

$$\pi_\ell^{\underline{p}}(f) = \pi_\ell(\Delta_{\underline{p}}\Delta^{-1/2}f), \tag{3.24}$$

acting boundedly on the space $L^{\underline{p}}(G/P,\ell)$. More generally, we have

Proposition 3.1 *Let $\underline{p},\underline{q}\in[1,\infty[^{1+d}$, and let $f\in L^1(G)$ such that $\Delta_{\underline{p}}\Delta_{\underline{q}}^{-1}f\in L^1(G)$. Then the operator $\pi_\ell^{\underline{p}}(f)$ extends uniquely from $L^{\underline{q}}(G/P,\ell)\cap L^{\underline{p}}(G/P,\ell)$ to a bounded operator on $L^{\underline{q}}(G/P,\ell)$, given by the formula*

$$\pi_\ell^{\underline{p}}(f)=\pi_\ell^{\underline{q}}(\Delta_{\underline{p}}\Delta_{\underline{q}}^{-1}f). \tag{3.25}$$

In particular, one has

$$||\pi_\ell^{\underline{p}}(f)||_{L^{\underline{q}}(G/P,\ell)\to L^{\underline{q}}(G/P,\ell)}\leq||\Delta_{\underline{p}}\Delta_{\underline{q}}^{-1}f||_1. \tag{3.26}$$

4 Auxiliary results

4.1 Compact operators acting on mixed L^p-spaces

In this subsection, let $M:=X\times Y$ be a product of two measure spaces (X,dx) and (Y,dy). For $1\leq p<\infty$, denote by $L^{\underline{p}}$ the mixed L^p-space $L^p(X,L^2(Y))$, endowed with the norm

$$||f||_{\underline{p}}:=\Big(\int_X(\int_Y|f(x,y)|^2\,dy)^{p/2}\,dx\Big)^{1/p}.$$

By $\mathbb{I}_A$ we denote the indicator function of a set A.

Lemma 4.1 *(i) Let $E=\{E_1,\dots,E_n\}$ be a family of disjoint measurable subsets of finite measure in X, and denote by $S:L^{\underline{p}}\to L^{\underline{p}}$ the associated averaging operator*

$$S(f)(x,y):=\sum_{j=1}^n\frac{1}{|E_j|}\left(\int_{E_j}f(u,y)\,du\right)\mathbb{I}_{E_j}(x),\quad f\in L^{\underline{p}},\ (x,y)\in X\times Y$$

with respect to the first variable. Then the operator norm of T is bounded by 1, for every p.

(ii) Similarly, let $\mathcal{F}=\{F_1,\dots,F_m\}$ be a family of disjoint measurable subsets of finite measure in Y, and denote by $T:L^{\underline{p}}\to L^{\underline{p}}$ the associated averaging operator

$$T(f)(x,y):=\sum_{j=1}^m\frac{1}{|F_j|}\left(\int_{F_j}f(x,v)\,dv\right)\mathbb{I}_{F_j}(y),\quad f\in L^{\underline{p}},\ (x,y)\in X\times Y$$

with respect to the second variable. Then the operator norm of T is bounded by 1, for every p.

Proof. In order to prove (i), we write $f_x(y):=f(x,y)$. Then

$$S(f)_x=\sum_j\frac{1}{|E_j|}\mathbb{I}_{E_j}(x)f_j,$$

where $f_j:=\int_{E_j}f_u\,du\in L^2(Y)$. Therefore

$$||S(f)_x||_2=\sum_j\frac{1}{|E_j|}||f_j||_2\mathbb{I}_{E_j}(x),$$

hence

$$\int_X||S(f)_x||_2^p\,dx=\sum_j\frac{1}{|E_j|^p}||f_j||_2^p|E_j|=\sum_j|E_j|^{1-p}||f_j||_2^p.$$

But, by Minkowski's integral inequality and Hölder's inequality,

$$\begin{aligned}||f_j||_2^p &= ||\int_{E_j} f_u\,du||_2^p \leq \left(\int_{E_j} ||f_u||_2\,du\right)^p \\ &\leq |E_j|^{p/p'} \int_{E_j} ||f_u||_2^p\,du = |E_j|^{p-1} \int_{E_j} ||f_u||_2^p\,du.\end{aligned}$$

Consequently,

$$||S(f)||_{\underline{p}}^p \leq \sum_j \int_{E_j} ||f_u||_2^p\,du = \int_X ||f_u||_2^p\,du = ||f||_{\underline{p}}^p.$$

The proof of (ii) is even simpler. Indeed, for $f \in L^{\underline{p}}$, we have

$$\begin{aligned}||T(f)||_{\underline{p}}^p &= \int_X \left(\int_Y \left|\sum_j \frac{1}{|F_j|}\mathbb{I}_{F_j}(y)\int_{F_j} f(x,v)\,dv\right|^2 dy\right)^{p/2} dx \\ &= \int_X \left(\sum_j \frac{1}{|F_j|}\left|\int_{F_j} f(x,v)\,dv\right|^2\right)^{p/2} dx \leq \int_X \left(\sum_j \int_{F_j} |f(x,v)|^2\,dv\right)^{p/2} dx \\ &\leq \int_X \left(\int_Y |f(x,v)|^2\,dv\right)^{p/2} dx = ||f||_{\underline{p}}^p.\end{aligned}$$

Q.E.D.

We are now in a position to prove the following variant for mixed L^p-spaces of an interpolation theorem by Krasnoselskii [12].

Theorem 4.2 *Let $p, q \in [1,\infty[,\ p \neq q$, and let K be a linear operator on $L^{\underline{p}} + L^{\underline{q}}$ which maps $L^{\underline{p}}$ compactly into $L^{\underline{p}}$ and $L^{\underline{q}}$ boundedly into $L^{\underline{q}}$. Then K is a compact operator from $L^{\underline{r}}$ to $L^{\underline{r}}$, for every r lying strictly between p and q.*

Proof. Observe that the space $\mathcal{S}$ of simple functions of the form $\sum_j \alpha_j \mathbb{I}_{E_j} \otimes \mathbb{I}_{F_j}$, where the E_j and F_j form finite families of measurable subsets of finite measure in X, respectively Y, lies dense in $L^{\underline{r}}$, for every $1 \leq r < \infty$.

Denote by $B_1(0)$ the unit ball centered at the origin in $L^{\underline{p}}$. Since $\mathcal{K} := \overline{K(B_1(0))}$ is a compact subset of $L^{\underline{p}}$, for every $k \in \mathbb{N}$ we may thus find simple functions $f_1^k, \ldots, f_{j_k}^k$ in $\mathcal{S}$ such that, for any $g \in \mathcal{K}$, there exists a j such that $||g - f_j^k||_{\underline{p}} < \frac{1}{k+1}$.

Choose next finite families $\mathcal{E}_k = \{E_1^k, \ldots, E_{n_k}^k\}$, respectively $\mathcal{F}_k = \{F_1^k, \ldots, F_{m_k}^k\}$ of disjoint measurable subsets of X, respectively of Y, such that every $f = f_j^k$ can be written as a linear combination of functions of the form $\mathbb{I}_{E_r^k} \otimes \mathbb{I}_{F_s^k}$.

Denote by S_k and T_k the averaging operator associated to $\mathcal{E}_k$, respectively $\mathcal{F}_k$ in Lemma 4.1, and let R_k be the operator of finite rank given by

$$R_k := S_k \circ T_k, \quad k \in \mathbb{N}.$$

Then $R_k f_j^k = f_j^k$ for any j, and so, if $g \in \mathcal{K}$, and if we choose j such that $||g - f_j^k||_{\underline{p}} < \frac{1}{k+1}$, then we obtain from Lemma 4.1 that

$$||g - R_k g||_{\underline{p}} \leq ||g - f_j^k||_{\underline{p}} + ||R_k(f_j^k - g)||_{\underline{p}} \leq \frac{2}{k+1}.$$

This shows that $\lim_{k\to\infty} R_k(g) = g$ in $L^{\underline{p}}$ for every $g \in \mathcal{K}$. As a consequence, we obtain

$$\lim_{k\to\infty} ||R_k \circ K - K||_{L^{\underline{p}} \to L^{\underline{p}}} = 0. \tag{4.1}$$

For, otherwise we could find a sequence of functions f_k in $B_1(0)$ and $\varepsilon > 0$, such that

$$||f_k||_{\underline{p}} = 1 \quad \text{and} \quad ||(R_k - \mathbb{I}) \circ K f_k||_{\underline{p}} \geq \varepsilon \quad \forall k. \tag{4.2}$$

Passing to a subsequence, if necessary, we could then assume that the sequence of functions $g_k := Kf_k$ had a limit g in $\mathcal{K}$, since $\mathcal{K}$ is compact. This would imply $\lim_{k\to\infty}(R_k - \mathbb{I})(g_k - g) = 0$, hence $\lim_{k\to\infty}(R_k - \mathbb{I}) \circ K(f_k) = 0$, contradicting (4.2).

On the other hand

$$||R_k \circ K - K||_{L^{\underline{q}} \to L^{\underline{q}}} \leq 2||K||_{L^{\underline{q}} \to L^{\underline{q}}}, \quad \forall k \in \mathbb{N}, \tag{4.3}$$

because $||R_k||_{L^q \to L^q}$ is bounded by 1 for every k. Applying the Riesz-Thorin interpolation theorem, it follows from (4.1) and (4.2) that

$$\lim_{k\to\infty} ||R_k \circ K - K||_{L^{\underline{r}} \to L^{\underline{r}}} = 0,$$

for every r lying strictly between p and q. Our assertion follows.

Q.E.D.

In the sequel, we shall denote the space of compact operators on a Banach space E by $\mathcal{K}(E)$. Moreover, if K is an operator as in Thm. 4.2, and if we consider K as a compact operator from $L^{\underline{r}}$ to $L^{\underline{r}}$, for r lying strictly between p and q, then we shall also write K_r in place of K. The spectrum of $K_r \in \mathcal{K}(L^{\underline{r}})$ will be denoted by $\sigma_r(K)$. Given $p \in [1, \infty[$, we shall denote the conjugate exponent by p', i.e. $\frac{1}{p} + \frac{1}{p'} = 1$.

As for the spectra of K on different $L^{\underline{r}}$–spaces, we have

Proposition 4.3 *Let $1 < p_0 \leq 2$, and let K be a linear operator on $L^{\underline{p_0}} + L^{\underline{p_0'}}$, mapping the space $L^{\underline{p_0}}$, as well as the dual space $L^{\underline{p_0'}}$, compactly into itself, so that, by the preceding theorem, K is a compact operator on $L^{\underline{p}}$ for every $p \in [p_o, p_o']$. Assume further that K is self-adjoint on $L^{\underline{2}}$. Let $\lambda \in \sigma_{p_o}(K) \setminus \{0\}$. Then λ is real, and every generalized eigenvector of K_{p_o} associated to λ is in fact an eigenvector, lying in $\bigcap_{p\in[p_0,p_0']} L^{\underline{p}}$. In particular, all $L^{\underline{p}}$- spectra coincide, i.e. $\sigma_p(K) = \sigma_2(K)$, and so do the eigenspaces corresponding to non-zero eigenvalues, for every $p \in [p_0, p_0']$.*

Proof. Let $\lambda \in \sigma_{p_o}(K)\setminus\{0\}$, and let $E \subset D$ be two compact neighbourhoods of the line segment $[\lambda, \overline{\lambda}]$, which are invariant under complex conjugation, such that D is also a neighbourhood of E. Since the non-zero eigenvalues are isolated, by shrinking E and D, if necessary, we may assume that $\sigma_q(K) \cap D \subset [\lambda, \overline{\lambda}]$, for $q = p_0, 2, p_0'$. Then for any $\mu \in D \setminus E^\circ$, we have that $(K - \mu)^{-1}$ exists on $L^{\underline{q}}$ for $q = p_0, 2, p_0'$, and there exists a constant $C > 0$, such that

$$||(K - \mu)^{-1}||_{L^{\underline{p_0}} \to L^{\underline{p_0}}} + ||(K - \overline{\mu})^{-1}||_{L^{\underline{p_0'}} \to L^{\underline{p_0'}}}|| \leq C, \quad \forall \mu \in D \setminus E^\circ.$$

Thus, by interpolation, we obtain

$$||(K - \mu)^{-1}||_{L^{\underline{p}} \to L^{\underline{p}}} \leq C, \text{ for every } \mu \in D \setminus E^\circ, \ p \in [p_0, p_0']. \tag{4.4}$$

In particular $(K_p - \mu)^{-1}$ exists on $L^{\underline{p}}$. This implies that no point $\mu \in D \setminus [\lambda, \overline{\lambda}]$ lies in any of the sets $\sigma_p(K)$, $p \in [p_0, p_0']$, i.e. $\sigma_p(K) \cap D \subset [\lambda, \overline{\lambda}]$. Let

$$\mathcal{D} := \bigcap_{p\in[p_0,p_0']} L^{\underline{p}}.$$

Then $\mathcal{D}$ is dense in $L^{\underline{p}}$ and invariant under K_p, for every $p \in [p_0, p_0']$.

We show that, for $\mu \in D \setminus [\lambda, \overline{\lambda}]$, the restriction

$$(K_p - \mu)^{-1}|_{\mathcal{D}}$$

of $(K_p - \mu)^{-1}$ to the joint core $\mathcal{D}$ does not depend on $p \in [p_0, p_0']$.

Indeed, let $\xi \in L^{\underline{p}} \cap L^{\underline{q}}$, and let

$$\eta_p := (K_p - \mu)^{-1}\xi, \quad \eta_q := (K_q - \mu)^{-1}\xi,$$

for given $p, q \in [p_0, p_0']$. Then, for $\varphi \in \mathcal{D}$, we have that

$$\langle \eta_p, (K - \overline{\mu})\varphi \rangle = \langle (K_p - \mu)^{-1}\xi, (K - \overline{\mu})\varphi \rangle = \langle \xi, \varphi \rangle = \langle \eta_q, (K - \overline{\mu})\varphi \rangle.$$

Since $(K - \mu)$ is invertible on $L^{\underline{p}}$, it follows that $(K - \overline{\mu})$ is invertible on the dual space $(L^{\underline{p}})'$, and the same applies to $(L^{\underline{q}})'$. Consequently, $(K - \overline{\mu})(\mathcal{D})$ is dense in $(L^{\underline{p}})'$ as well as in $(L^{\underline{q}})'$, so that η_p and η_q coincide as linear functionals on $(L^{\underline{p}})'$ and on $(L^{\underline{q}})'$, and so $\eta_p = \eta_q \in L^{\underline{p}} \cap L^{\underline{q}}$. This implies that $(K_q - \mu)^{-1}\xi = (K_{p_0} - \mu)^{-1}\xi$ for every $\xi \in \mathcal{D}$ and every $q \in [p_0, p_0']$.

Let Γ be a simple curve in $D \setminus E^\circ$ whose winding number with respect to every point in the interval $[\lambda, \overline{\lambda}]$ is one, and let

$$P_p := \int_\Gamma (K_p - \gamma)^{-1}\, d\gamma, \quad p \in [p_0, p_0'].$$

Then the operators P_p coincide on $\mathcal{D}$ and are bounded by a common constant, by (4.4). The image $\operatorname{im} P_p$ of P_p is the sum of the generalized eigenspaces of K_p over all eigenvalues of K_p which are contained in $\sigma_p \cap [\lambda, \overline{\lambda}]$. In particular, the image of P_p is finite dimensional, since K_p is compact. Since $\operatorname{im} P_p = \overline{P_p(\mathcal{D})}$, we thus have

$$\operatorname{im} P_p = P_p\Big(\bigcap_{q \in [p_0, p_0']} L^{\underline{q}}\Big) = \operatorname{im} P_2.$$

In particular, $\operatorname{im} P_p$ does not depend on p.

Since K is self-adjoint on $L^{\underline{2}}$, its restriction to $\operatorname{im} P_2$ is also self-adjoint on $\operatorname{im} P_2$, and so every eigenvalue of K, when acting on $\operatorname{im} P_p$, is real, for any $p \in [p_0, p_0']$. This applies in particular to λ. We thus see that $\sigma_p(K) \cap D = \{\lambda\}$, that $\operatorname{im} P_2 \subset \mathcal{D}$, and that $\operatorname{im} P_2$ is in fact the eigenspace of K associated to λ, for any $p \in [p_0, p_0']$. Furthermore, every eigenvector of K for the eigenvalue λ in $L^{\underline{p}}$ is contained in $\bigcap_{q \in [p_0, p_0']} L^{\underline{q}}$.

Q.E.D.

4.2 Approximate units of Herz-Schur multipliers

Let G be a locally compact group, endowed with a left-invariant Haar measure dx. If K is a continuous function on G, we shall denote by $\lambda(K) : C_0(G) \to C(G)$ the convolution operator given by

$$\lambda(K)(\varphi) := K \star \varphi, \quad \varphi \in C_0(G);$$

here, λ denotes again the left-regular representation. In case that $\lambda(K)$ extends to a bounded operator on $L^p(G)$, for some $p \in [1, \infty[$, we shall denote the (unique) extension also by $\lambda(K)$.

The following result is a consequence of the well-known theory of Herz-Schur multipliers and Fourier-Figa-Talamanca-Herz-Eymard algebras, see e.g. [9], [5], [6]. For the convenience of the reader, we shall provide a proof.

Lemma 4.4 *Let $K \in C(G)$, and assume that $\lambda(K)$ is bounded on $L^p(G)$. Let $\psi \in L^{p'}(G)$ and $\eta \in L^p(G)$. Then also $\lambda((\psi \star \check{\eta})K)$ is bounded on $L^p(G)$, and*

$$||\lambda((\psi \star \check{\eta})K)||_{L^p \to L^p} \le ||\psi||_{p'} ||\eta||_p ||\lambda(K)||_{L^p \to L^p}. \tag{4.5}$$

Proof. If $f, g \in C_0(G)$, then

$$\begin{aligned}
\langle \lambda((\psi \star \check{\eta})K)f, g\rangle &= \int\int K(y)(\psi \star \check{\eta})(y) f(y^{-1}x) g(x)\, dy\, dx \\
&= \int\int\int K(y)\psi(t)\check{\eta}(t^{-1}y) f(y^{-1}x) g(x)\, dy\, dx\, dt \\
&= \int\int\int K(y)\psi(xt)\eta(y^{-1}xt) f(y^{-1}x) g(x)\, dy\, dx\, dt \\
&= \int \langle K \star (\eta_t f), \psi_t g\rangle\, dt,
\end{aligned}$$

where we have used the abbreviation $h_t(x) := h(xt)$. Thus, by Hölder's inequality and Fubini's theorem, we obtain

$$\begin{aligned}
&|\langle \lambda((\psi \star \check{\eta})K)f, g\rangle| \\
\le\ & \int ||\lambda(K)||_{L^p \to L^p} ||\eta_t f||_p ||\psi_t g||_{p'}\, dt \\
\le\ & ||\lambda(K)||_{L^p \to L^p} \Big(\int\int |\eta(yt)|^p |f(y)|^p\, dy\, dt\Big)^{1/p} \Big(\int\int |\psi(xt)|^{p'} |g(x)|^{p'}\, dx\, dt\Big)^{1/p'} \\
=\ & ||\lambda(K)||_{L^p \to L^p} ||\eta||_p ||\psi||_{p'} ||f||_p ||g||_{p'}.
\end{aligned}$$

This implies (4.5).

Q.E.D.

Assume next that G is a Lie group. If T is a right-invariant bounded operator on $L^p(G)$, then it follows from the Schwartz' kernel theorem that there exists a unique distribution $K \in \mathcal{D}'(G)$, such that

$$T\varphi = K \star \varphi = \lambda(K)(\varphi), \quad \text{for every } \varphi \in \mathcal{D}(G). \tag{4.6}$$

We shall then also denote K by $T\delta_e$. Inversely, given a distribution $K \in \mathcal{D}'(G)$, we shall denote by $\lambda(K) : \mathcal{D}(G) \to C^\infty(G)$ the convolution operator given by

$$\lambda(K)(\varphi) := K \star \varphi, \quad \varphi \in \mathcal{D}(G).$$

In case that $\lambda(K)$ extends to a bounded operator on $L^p(G)$, for some $p \in [1, \infty[$, we shall again denote also the (unique) extension by $\lambda(K)$. The following proposition makes use of ideas in [13], [5].

Proposition 4.5 *Let G be an amenable Lie group, let $1 \le p < \infty$, and let $K \in \mathcal{D}'(G)$, such that the convolution operator $\lambda(K)$ is bounded on $L^p(G)$. Assume further that $\lambda(K) \in C^*(G)$. Then $\lambda(K)$ is bounded on $L^q(G)$ for every exponent q lying between 2 and p, and there exists a sequence $\{K_n\}_n$ in $C_0(G)$, such that*

$$\lim_{n\to\infty} ||\lambda(K) - \lambda(K_n)||_{L^2 \to L^2} = 0, \tag{4.7}$$

and

$$||\lambda(K_n)||_{L^q \to L^q} \le ||\lambda(K)||_{L^q \to L^q} \quad \forall n \in \mathbb{N}, \tag{4.8}$$

for every q lying between 2 and p.

Proof. Assume first that $K \in C(G)$. We choose an increasing sequence $\{A_n\}_{n\in\mathbb{N}}$ of compact subsets of G such that $G = \bigcup_{n\in\mathbb{N}} A_n$. Let $\varepsilon > 0$. Since G is amenable, we can find compact subsets U_n of G such that

$$\frac{|(A_n^{-1}U_n) \triangle U_n|}{|U_n|} < \varepsilon.$$

Put

$$\phi_{n,\varepsilon} := \frac{1}{|U_n|} \mathbb{I}_{U_n} \star \check{\mathbb{I}}_{U_n} \in C_0(G).$$

For any $q \in [1, \infty[$, let us put

$$\psi_{n,\varepsilon} = \frac{1}{|U_n|^{1/p'}} \mathbb{I}_{U_n}, \quad \eta_{n,\varepsilon} := \frac{1}{|U_n|^{1/p}} \mathbb{I}_{U_n}, \quad n \in \mathbb{N}.$$

Then of course

$$||\psi_{n,\varepsilon}||_{p'} = 1 = ||\eta_{n,\varepsilon}||_p, \quad n \in \mathbb{N},$$

and

$$\phi_{n,\varepsilon} = \psi_{n,\varepsilon} \star \check{\eta}_{n,\varepsilon}.$$

If $a \in A_n$, then

$$\begin{aligned} ||\lambda(a)\eta_{n,\varepsilon} - \eta_{n,\varepsilon}||_p^p &= \frac{1}{|U_n|} \int_G \left| \mathbb{I}_{U_n}(a^{-1}t) - \mathbb{I}_{U_n}(t) \right|^p dt \\ &= \frac{1}{|U_n|} \int_G \mathbb{I}_{(aU_n)\triangle U_n}(t)\, dt \le \varepsilon. \end{aligned}$$

Observe that $\phi_{n,\varepsilon}(e) = 1$. Therefore, for any $a \in A_n$, we have that

$$\begin{aligned} |\phi_{n,\varepsilon}(a) - 1| &= |\phi_{n,\varepsilon}(a) - \phi_{n,\varepsilon}(e)| = |\int \psi_{n,\varepsilon}(t)[\eta_{n,\varepsilon}(a^{-1}t) - \eta_{n,\varepsilon}]\, dt| \\ &\le ||\psi_{n,\varepsilon}||_{p'} ||\lambda(a)\eta_{n,\varepsilon} - \eta_{n,\varepsilon}||_p \le \varepsilon^{1/p}. \end{aligned}$$

Thus, if we put $\phi_n := \phi_{n,1/n},\ n \ge 1$, then we see that the functions ϕ_n tend to 1, uniformly on compacta. Let us put $K_n := \phi_n K$. Then $K_n \in C_0(G)$, and, by the preceding proposition, the K_n satisfy (4.8). Moreover, since $\lambda(K) \in C^*(G)$, there exists a sequence $\{f_j\}_j$ in $C_0(G)$ such that $\lim_{j\to\infty} ||\lambda(K) - \lambda(f_j)|| = 0$, where by $||\cdot||$ we denote the operator norm on $L^2(G)$. Since $\lim_{n\to\infty} ||f_j - \phi_n f_j||_1 = 0$, for every f_j we see that $\lim_{n\to\infty} ||\lambda(f_j) - \lambda(\phi_n f_j)|| = 0$. Moreover, by Lemma 4.4, we have

$$\begin{aligned} ||\lambda(K) - \lambda(K_n)|| &\le ||\lambda(K) - \lambda(f_j)|| + ||\lambda(f_j) - \lambda(\phi_n f_j)|| + ||\lambda(\phi_n(f_j - K))|| \\ \le 2 &\quad ||\lambda(K) - \lambda(f_j)|| + ||\lambda(f_j) - \lambda(\phi_n f_j)||. \end{aligned}$$

Thus, given $\varepsilon > 0$, we may first choose j such that $||\lambda(K) - \lambda(f_j)|| < \varepsilon/2$, and then $n_0 \in \mathbb{N}^\times$ such that $||\lambda(f_j) - \lambda(\phi_n f_j)|| < \varepsilon/4$ for every $n \ge n_0$. Then $||\lambda(K) - \lambda(K_n)|| < \varepsilon$ for every $n \ge n_0$, i.e. (4.7) is also satisfied.

For an arbitrary $K \in \mathcal{D}'(G)$ satisfying the assumptions of Prop. 4.5, we may argue as follows. We fix an approximate identity $\{\chi_i\}_i$ in $\mathcal{D}(G)$ such that $||\chi_i||_1 = 1$ for every i, and put $K^{(i)} := \chi_i \star K$. Then $K^{(i)} \in C(G)$. Moreover,

$$\begin{aligned} ||\lambda(K) - \lambda(K^{(i)})|| &\le ||\lambda(K) - \lambda(f_j)|| + ||\lambda(f_j) - \lambda(\chi_i \star f_j)|| + ||\lambda(\chi_i \star (f_j - K))|| \\ \le 2 &\quad ||\lambda(K) - \lambda(f_j)|| + ||f_j - \chi_i \star f_j||_1, \end{aligned}$$

where $\{f_j\}_j$ is again a sequence in $C_0(G)$ such that $\lim_{j\to\infty}||\lambda(K)-\lambda(f_j)||=0$. Thus, arguing similarly as before, we see that

$$\lim_{i\to\infty}||\lambda(K)-\lambda(K^{(i)})||=0.$$

Let us put $K_\nu^{(i)}:=\phi_\nu K^{(i)}\in C_0(G)$. Then, we know already that

$$||\lambda(K_\nu^{(i)})||_{L^q\to L^q}\le||\lambda(K^{(i)})||_{L^q\to L^q}\le||\chi_i||_1||\lambda(K)||_{L^q\to L^q}=||\lambda(K)||_{L^q\to L^q}.$$

Moreover, we have

$$||\lambda(K)-\lambda(K_\nu^{(i)})||\le||\lambda(K)-\lambda(K^{(i)})||+||\lambda(K^{(i)})-\lambda(K_\nu^{(i)})||.$$

Thus, if $\varepsilon>0$ is given, we may choose i such that $||\lambda(K)-\lambda(K^{(i)})||<\varepsilon/2$, and subsequently $n_0\in\mathbb{N}^\times$ such that $||\lambda(K^{(i)})-\lambda(K_\nu^{(i)})||<\varepsilon/2$ for every $\nu\ge n_0$. Then $||\lambda(K)-\lambda(K_\nu^{(i)})||<\varepsilon$ for every $\nu\ge n_0$. This shows that we may find a subsequence K_n among the $K_\nu^{(i)}$ satisfying (4.7) and (4.8).

Q.E.D.

5 An analytic family of compact operators

Let us now choose a fixed sub-Laplacian L on G, and denote by $\{e^{-tL}\}_{t>0}$ the heat semigroup generated by L. We recall some well-known facts about this semi-group (see e.g. [15], also for further references).

For every $t>0$, e^{-tL} is a convolution operator

$$e^{-tL}f=h_t\star f, \tag{5.1}$$

where the $\{h_t\}_{t>0}$ form a 1-parameter semigroup of smooth probability measures in $L^1(G)$.

Moreover, as a consequence of Gaussian estimates for the heat kernels, one has the following extension of Lemma 5.1 in [15], whose proof carries over to the present situation.

Proposition 5.1 *Let $\mathfrak{s}$ be a subspace of $\mathfrak{g}$ complementary to $\mathfrak{n}$, for instance $\mathfrak{s}=\mathfrak{a}+\mathfrak{b}$, so that the mapping $\mathfrak{s}\times N\ni(S,n)\mapsto\exp(S)n\in G$ is a diffeomorphism from $\mathfrak{s}\times N$ onto G, and fix a norm $|\cdot|$ on $\mathfrak{s}$. For any $a\ge0, j\in\mathbb{N}$, put*

$$h_1^{a,j}(\exp(S)n):=|S|^je^{a|S|}h_1(\exp(S)n),\qquad (S,n)\in\mathfrak{s}\times N.$$

Then $h_{a,j}\in L^1(G)$. Moreover, there is a constant $C_a>0$, such that

$$||h_1^{a,j}||_1\le C_a^{j+1}\Gamma\left(\frac{j}{2}+1\right). \tag{5.2}$$

If χ is any continuous, real or complex character of G, with differential $d\chi\in\mathfrak{g}_{\mathbb{C}}^*$, then $\chi(\exp(S)n)=e^{d\chi(S)}$. We therefore have the following

Corollary 5.2 *$\chi h_1\in L^1(G)$ for every continuous character χ of G.*

From now on, we shall make the following

Assumption. $\ell \in \mathfrak{g}^*$ satisfies Boidol's condition, and $\Omega(\ell)|_{\mathfrak{n}}$ is closed.
Moreover, we assume and $p \in [1, \infty[,\ p \neq 2$, is fixed.

Then, since ℓ satisfies (B), there exists at least one root λ of $\mathfrak{g}$, such that $\lambda|_{\mathfrak{g}(\ell)}$ is a non-trivial root of the $\mathfrak{g}(\ell)$-module $\mathfrak{g}/\mathfrak{p}$ (see [2]). Consequently, there exists at least one index $i \in \{1, \ldots, d\}$, such that

$$\lambda_i|_{\mathfrak{g}(\ell)} \neq 0,$$

and $(\mathfrak{g}_{j_i} + \mathfrak{p})/(\mathfrak{g}_{j_i+1} + \mathfrak{p}) \neq \{0\}$. Notice that the latter condition is equivalent to $\varepsilon_{j_i} \neq 0$. Choose i minimal with these properties, and put

$$\overline{p} := (p, \ldots, p, 2, 2, \ldots, 2), \tag{5.3}$$

where the last p occurs at the i-th position.

Then, for $T \in \mathfrak{p}$, we have

$$(\delta_{\overline{p}} - \delta_{\overline{2}})(T) = \sum_{k=1}^{i} \frac{1}{p} \varepsilon_{j_k} \tau_{j_k}(T) + \sum_{k=i+1}^{d} \frac{1}{2} \varepsilon_{j_k} \tau_{j_k}(T) - \sum_{k=1}^{d} \frac{1}{2} \varepsilon_{j_k} \tau_{j_k}(T) = (\frac{1}{p} - \frac{1}{2}) \sum_{k=1}^{i} \varepsilon_{j_k} \tau_{j_k}(T),$$

and for $X \in \mathfrak{a} + \mathfrak{n}$ one has

$$(\delta_{\overline{p}} - \delta_{\overline{2}})(X) = (\frac{1}{p} - \frac{1}{2}) \sum_{k=1}^{m-1} \varepsilon_k \tau_k(X).$$

Thus, if we define the real character ν of $\mathfrak{g}$ by

$$\nu(X) := \sum_{k=1}^{i} \varepsilon_{j_k} \tau_{j_k}(X), \quad \text{if } X \in \mathfrak{p},$$

$$\nu(X) := \sum_{k=1}^{m-1} \varepsilon_k \tau_k(X), \quad \text{if } X \in \mathfrak{a} + \mathfrak{n},$$

then

$$\Delta_{\overline{p}} \Delta_{\overline{2}}^{-1}(\exp(X)) = e^{(\frac{1}{2} - \frac{1}{p})\nu(X)}, \quad X \in \mathfrak{g}. \tag{5.4}$$

Moreover, since $\tau_{j_k}(T) = 0$ for $1 \leq k < i$ and $T \in \mathfrak{g}(\ell)$, we have

$$\nu|_{\mathfrak{g}(\ell)} = \varepsilon_{j_i} \tau_{j_i}|_{\mathfrak{g}(\ell)} \neq 0. \tag{5.5}$$

For any complex number z in the strip

$$\Sigma := \{\zeta \in \mathbb{C} : |\mathrm{Im}\, \zeta| < 1/2\},$$

let Δ_z be the complex character of G given by

$$\Delta_z(\exp(X)) := e^{-iz\nu(X)}, \quad X \in \mathfrak{g},$$

and χ_z the unitary character

$$\chi_z(\exp(X)) := e^{-i\mathrm{Re}\,(z)\nu(X)}, \quad X \in \mathfrak{g}.$$

Since, by (5.4),

$$\Delta_z(x) = \chi_z \Delta_{\overline{p(z)}} \Delta_{\overline{2}}^{-1},$$

if we define $p(z) \in]1, \infty[$ by the equation

$$\mathrm{Im}\,(z) = 1/2 - 1/p(z), \tag{5.6}$$

we see that the representation π_ℓ^z, given by

$$\pi_\ell^z(x) := \Delta_z(x)\pi_\ell(x) = \chi_z(x)\pi_\ell^{\overline{p(z)}}(x), \quad x \in G, \tag{5.7}$$

is an isometric representation on the space $L^{\overline{p(z)}}(G/P, \ell)$.

Observe that for $\tau \in \mathbb{R}$, we have $p(\tau) = 2$, and $\pi_\ell^\tau = \chi_\tau \otimes \pi_\ell$ is a unitary representation on $L^{\overline{2}}$. Moreover,

$$\pi_\ell^\tau \simeq \pi_{\ell-\tau\nu}, \tag{5.8}$$

since the mapping $f \mapsto \overline{\chi}_\tau f$ intertwines the representations $\chi_\tau \otimes \pi_\ell$ and $\pi_{\ell-\tau\nu}$.

Let us put

$$T(z) := \pi_\ell^z(h_1) = \pi_\ell(\Delta_z h_1), \quad z \in \Sigma,$$

and let us shortly write

$$L^{\overline{p}} := L^{\overline{p}}(G/P, \ell), \quad 1 \le p < \infty,$$

where $\overline{p}$ is given by (5.3).

Since, by (3.24),

$$T(z) = \pi_\ell^{\overline{q}}(\Delta_z \Delta_{\overline{2}} \Delta_{\overline{q}}^{-1} h_1), \tag{5.9}$$

it follows from Cor.5.2 and Prop. 3.1 that the operator $T(z)$ leaves $L^{\overline{q}}$ invariant for every $1 \le q < \infty$, and is bounded on all these spaces. Much more is even true. Let us write $T_q(z)$ in place of $T(z)$, if we consider $T(z)$ as a bounded operator on $L^{\overline{q}}$. The spectrum of $T_q(z)$ will be denoted by $\sigma_q(z)$.

Proposition 5.3 *For every $q \in]1, \infty[$, the mapping $\Sigma \ni z \mapsto T_q(z)$ is an analytic family of compact operators in the sense of Kato ([11]). Moreover, if $\tau \in \mathbb{R}$, then $T_2(\tau)$ is self-adjoint on $L^{\overline{2}}$.*

Proof. By Thm. 2.2, the orbit $\Omega(\ell)$ is closed. Moreover, Cor. 5.2 shows that $\Delta_z h_1 \in L^1(G)$, and consequently $T(z) = \pi_\ell(\Delta_z h_1)$ is a compact operator on $L^{\overline{2}}$. On the other hand, in a similar way we see from (5.9) that $T(z)$ is bounded on $L^{\overline{q}}$, for every $1 \le q < \infty$. Since $L^{\overline{q}}$ is a mixed L^p-space of the type $L^p(X, L^2(Y))$ considered in Section 4.1, we may apply Thm. 4.2 to conclude that $T_q(z)$ is compact for every $q \in]1, \infty[$.

Next, let $\zeta \in \Sigma$ be fixed, and consider $T(\zeta + z)$, for $|z|$ sufficiently small. We have

$$T(\zeta + z) = \pi_\ell(\Delta_{\zeta+z} h_1) = \sum_{j=0}^{\infty} \frac{(-iz)^j}{j!} S_j, \tag{5.10}$$

where

$$S_j := \pi_\ell((\nu \circ \log)^j \Delta_\zeta h_1) = \pi_\ell^{\overline{q}}((\nu \circ \log)^j \Delta_\zeta \Delta_{\overline{2}} \Delta_{\overline{q}}^{-1} h_1).$$

By Prop. 5.1 we see that

$$||S_j||_{L^{\overline{q}} \to L^{\overline{q}}} \le C_q^{j+1} \Gamma(\frac{j}{2} + 1), \tag{5.11}$$

where the constant $C_q > 0$ stays bounded whenever q runs through a compact interval. Moreover, arguing for S_j as we did for $T(z)$ before, one finds that $S_j \in \mathcal{K}(L^{\overline{q}})$, for every $q \in]1, \infty[$. Thus, by (5.10) and (5.11), the mapping $z \mapsto T_q(z)$ is holomorphic from Σ into $\mathcal{K}(L^{\overline{q}})$, for $1 < q < \infty$ (it even extends to an entire mapping from $\mathbb{C}$ into $\mathcal{K}(L^{\overline{q}})$.)

Finally, if $\tau \in \mathbb{R}$, then π_ℓ^τ is a unitary representation on $L^{\overline{2}}$, so that $T(\tau)^* = \pi_\ell^\tau(h_1)^* = \pi_\ell^\tau(h_1^*) = \pi_\ell^\tau(h_1) = T(\tau)$.

Q.E.D.

We can now prove the following perturbation result.

Proposition 5.4 *Let $1 \le p_0 < 2$. There exist an open neighborhood U of a point $z_0 \in \mathbb{R}$ in the complex strip Σ and holomorphic mappings*

$$\lambda : U \to \mathbb{C}$$

and

$$\xi : U \to \bigcap_{p_0 \le p \le p_0'} L^{\overline{p}},$$

such that $\xi(z) \neq 0$ and

$$T(z)\xi(z) = \lambda(z)\xi(z) \quad \textit{for every} \quad z \in U. \tag{5.12}$$

Moreover, shrinking U, if necessary, one can find a constant $C > 0$ such that

$$||\xi(z)||_{L^{\overline{p}}} \le C \quad \textit{for every} \quad z \in U,\ p \in [p_0, p_0']. \tag{5.13}$$

Proof. Let $z_0 \in \Sigma$ be real. Then, by Prop. 5.3 and Prop 4.3, the $L^{\overline{p}}$- spectrum of $T_p(z_0)$ is independent of p, and agrees thus with $\sigma_2(z_0)$. Moreover, for every non-trivial eigenvalue $\lambda_0 \in \sigma_2(z_0)$ of $T(z_0)$, every generalized eigenvector ξ_0 associated to λ_0 is in fact an eigenvector, lying in $\mathcal{D} := \bigcap_{p_0 \le p \le p_0'} L^{\overline{p}}$. Let us for instance choose for λ_0 the largest eigenvalue of $T(z_0)$.

Consider the three analytic families

$$\{T_q(z)\}_{z\in\Sigma}, \quad \text{for} \quad q = p_0, 2, p_0'.$$

By choosing z_0 in such a way that λ_0 is a non-branching eigenvalue for all three analytic families and applying analytic perturbation theory (see [11]), we may find an open connected neighborhood U of some real point $z_0 \in \mathbb{R}$ in Σ and three holomorphic families $U \ni z \mapsto \lambda_q(z)$ of eigenvalues for the operator $T_q(z)$, $\quad q = p_0, 2, p_0'$, which all coincide at z_0 with the eigenvalue λ_0. Moreover, we may choose a neighborhood W of λ_0 such that

$$W \cap \sigma_q(z) = \{\lambda_q(z)\} \quad \text{for} \quad z \in U,\ q = p_0, 2, p_0'.$$

Let

$$P_q(z) := \int_\Gamma (T_q(z) - \mu)^{-1}\, d\mu, \quad z \in U,$$

where Γ is a circle in W winding around λ_0 once. By shrinking U, if necessary, we may also assume that the curve Γ separates $\lambda_q(z)$ from the remaining elements of $\sigma_q(z)$, for every $z \in U$ and $q = p_0, 2, p_0'$. Then $P_q(z)$ projects onto the generalized eigenspace $E_q(z)$ of $T_q(z)$ associated with the eigenvalue $\lambda_q(z)$, for $q = p_0, 2, p_0'$. Moreover, $\mathcal{D}$ is the core considered in the proof of Prop.4.3, and we had seen there that, for real $z \in U$, the restrictions of the resolvents $(T_p(z) - \mu)^{-1}$ to $\mathcal{D}$ do not depend on p, so that the same applies to the projectors $P_p(z)$. Since $P_q(z)$ depends holomorphically on z, it follows that

$$P_q(z)|_{\mathcal{D}} = P_2(z)|_{\mathcal{D}} \quad \text{for every} \quad z \in U \cap \mathbb{R},\ q = p_0, 2, p_0' \tag{5.14}$$

Moreover, since we may assume that $P_q(z)$ is uniformly bounded on $L^{\overline{q}}$, for $z \in U$ and $q = p_0, 2, p_0'$, by interpolation we derive from (5.14) that

$$||P_2(z)\xi||_{L^{\overline{p}}} \le C||\xi||_{L^{\overline{p}}} \quad \forall z \in U,\ p \in [p_0, p_0'],\ \xi \in \mathcal{D}. \tag{5.15}$$

Choose now $\xi_0 \in \mathcal{D} \setminus \{0\}$ such that $T(z_0)\xi_0 = \lambda_0 \xi_0$, and put

$$\xi(z) := P_2(z)\xi_0, \quad z \in U.$$

Then $\xi(z) \in \mathcal{D}\setminus\{0\}$ for every $z \in U$, (5.13) holds because of (5.15), and the mapping $z \mapsto \xi(z) \in \mathcal{D}$ is holomorphic with respect to every $L^{\overline{p}}$-norm, $p_0 \leq p \leq p_0'$, provided we choose U sufficiently small.

Furthermore, for real $z \in U$, we have $T(z)\xi(z) = \lambda(z)\xi(z)$, and since both sides of this equation depend holomorphically on z, it remains valid for every $z \in U$.

Q.E.D.

6 The proof of Theorem 1

We have to show that, under the assumptions made in the previous section, there exist a point λ_0 in the L^2-spectrum of L and an open neighborhood $\mathcal{U}$ of λ_0 in $\mathbb{C}$, such that every L^p-multiplier $F \in C_\infty(\mathbb{R})$ extends holomorphically to $\mathcal{U}$. Since $\sigma_2(L) = [0, \infty[=: \mathbb{R}_+$, we may restrict ourselves to multipliers on $\mathbb{R}_+$.

Lemma 6.1 *Let $1 \leq p < \infty$, and let $F \in \mathcal{M}_p(L) \cap C_\infty(\mathbb{R}_+)$. Then $F(L) \in C^*(G)$, and $F(L)$ is bounded on $L^q(G)$ for every q lying between p and p'. Moreover, there exists a constant $C \geq 0$, such that*

$$||F(L)||_{L^q \to L^q} \leq C, \quad \text{if} \quad |\frac{1}{q} - \frac{1}{2}| \leq |\frac{1}{p} - \frac{1}{2}|. \tag{6.1}$$

Proof. Since $F \in C_\infty(\mathbb{R}_+)$, there exists a sequence of functions of Laplace transform type $\tilde{\varphi}_n(\lambda) = \int_0^\infty \varphi_n(t) e^{-\lambda t}\, dt$, where $\varphi_n \in C_0(\mathbb{R}_+)$, which converges uniformly to F (see [15], Prop.2.1). Moreover, if φ is a real valued function on $\mathbb{R}_+$, then $\tilde{\varphi}$ is real valued too, as is the convolution kernel $\tilde{\varphi}(L)\delta_e = \int_0^\infty \varphi(t) h_t\, dt$ associated to $\tilde{\varphi}(L)$. Let $\alpha_n := \operatorname{Re} \varphi_n$, $\beta_n := \operatorname{Im} \varphi_n$, and $F_1 := \operatorname{Re} F, F_2 := \operatorname{Im} F$. Then F_1 is the uniform limit of the $\tilde{\alpha}_n$, and F_2 is the uniform limit of the $\tilde{\beta}_n$, so that consequently, for every real-valued function $f \in \mathcal{D}(G)$, one has

$$F_1(L)f = \lim_{n\to\infty} \tilde{\alpha}_n(L)f \text{ and } F_2(L)f = \lim_{n\to\infty} \tilde{\beta}_n(L)f$$

in $L^2(G)$. This shows that the $F_1(L)f$ and $F_2(L)f$ are real-valued functions, whence

$$||F(L)f||_p = \Big(\int_G (|[F_1(L)f](x)|^2 + |[F_2(L)f](x)|^2)^{p/2}\, dx\Big)^{1/p} \geq \max(||F_1(L)f||_p, ||F_2(L)f||_p).$$

Hence F_1 and F_2 are L^p-multipliers for L too, and since $F_1(L)^* = \overline{F}_1(L) = F_1(L)$ as well as $F_2(L)^* = F_2(L)$, we see that $F_1(L)$ and $F_2(L)$ are also bounded on $L^{p'}(G)$. This shows that $F(L) = F_1(L) + iF_2(L)$ is $L^{p'}$-bounded, and (6.1) follows by interpolation.

Q.E.D.

In view of Lemma 6.1, we may and shall assume in the sequel that $1 < p < 2$. Let $K := F(L)\delta_e$ be the convolution kernel of $F(L)$, so that $F(L)\varphi = K \star \varphi = \lambda(K)\varphi$, for $\varphi \in \mathcal{D}(G)$. According to Prop. 4.5, choose a sequence $\{K_n\}_n$ in $C_0(G)$ such that

$$\lim_{n\to\infty} ||\lambda(K) - \lambda(K_n)||_{L^2 \to L^2} = 0, \tag{6.2}$$

and

$$||\lambda(K_n)||_{L^q \to L^q} \leq ||\lambda(K)||_{L^q \to L^q} \quad \forall n \in \mathbb{N}, \tag{6.3}$$

for every q lying between 2 and p.

We now apply the transference theorem in order to conclude that

$$||\pi_\ell^z(K_n)||_{\overline{L^{p(z)}}\to\overline{L^{p(z)}}} \le ||\lambda(K_n)||_{L^{p(z)}(G)\to L^{p(z)}(G)} \quad \forall n\in\mathbb{N}, z\in\Sigma. \tag{6.4}$$

To this end, recall that our representation π_ℓ^z acts isometrically on a mixed L^p-space of the type $L^p(X, L^2(Y))$, with $p = p(z)$. Such a space can be embedded into an L^p-space. Namely, $L^2(Y)$ is isometrically isomorphic to a subspace of $L^p(Z)$ (see [16] Lemme 1 or [7] Corollary 1). Then of course $L^p(X, L^2(Y))$ is isometrically isomorphic to a subspace of $L^p(X\times Z)$. However, the proof of Theorem 2.4 in [3] remains valid also for bounded representations on closed subspaces of L^p-spaces, and we can thus apply the transference theorem in [3] to obtain (6.4).

Remark. (6.4) is an immediate consequence of the inclusion $A(\xi)A_p \subset A_p$ of Theorem A in [7] and Theorem 6 in [8]. It seems that Herz understood transference very well, but in his publications rather concentrated on abstract results (notably in [7]) and made explicit only few consequences. We felt that in case of (6.4) (as in section 4) re-proving certain more or less known results is more convenient then explaining how to translate and properly combine known results to get what we need.

From (6.1), (6.3) and (6.4) we get

$$||\pi_\ell^z(K_n)||_{\overline{L^{p(z)}}\to\overline{L^{p(z)}}} \le C \quad \text{for every} \quad z\in\Sigma_p, \tag{6.5}$$

where Σ_p denotes the smaller strip

$$\Sigma_p := \{\zeta\in\mathbb{C} : |\mathrm{Im}\,\zeta| < \frac{1}{p} - \frac{1}{2}\}.$$

Next, letting $p_0 := p$, choose holomorphic families $\lambda(z)$ of eigenvalues for $\pi_\ell^z = T(z)$ and associated eigenfunctions $\xi(z),\ z\in U$, as in Prop. 5.4, and assume w.r. that $U\subset\Sigma_p$.

Fix $\psi\in C_0^\infty(G/P,\ell)$ such that $\langle\xi(z),\psi\rangle \ne 0$ for every $z\in U$ (if necessary, we have to shrink U another time to achieve this), and consider the complex functions

$$h_n : U\to\mathbb{C}, \quad z\mapsto \langle\pi_\ell^z(K_n)\xi(z),\psi\rangle, \quad n\in\mathbb{N},$$

which are holomorphic in U.

By (6.5) and (5.13) we obtain

$$|h_n(z)| \le ||\pi_\ell^z(K_n)||_{\overline{L^{p(z)}}\to\overline{L^{p(z)}}}\, ||\xi(z)||_{\overline{p(z)}}\, ||\psi||_{\overline{p(z)'}} \le C, \quad \forall z\in U, n\in\mathbb{N}.$$

The family of functions $\{h_n\}_n$ is thus a normal family of holomorphic functions, and so, by the theorems of Montel and Weierstraß, there exists a subsequence $\{h_{n_j}\}_j$ which converges locally uniformly to a holomorphic limit function h on U.

Now, if $z\in U$ is real, then the representation π_ℓ^z is unitary, and since, by (6.2), $\lambda(K_n)$ converges to $\lambda(K)$ in $C^*(G)$, as n tends to ∞, it follows that $h_n(z)$ converges on $U\cap\mathbb{R}$ to $\langle\pi_\ell^z(K)\xi(z),\psi\rangle$, i.e.

$$h(z) = \langle\pi_\ell^z(K)\xi(z),\psi\rangle, \quad z\in U\cap\mathbb{R}. \tag{6.6}$$

Let $\mu(z) := -\log\lambda(z),\ z\in U$, where log denotes the principal branch of the logarithm. Then μ is holomorphic on U, and from $\pi_\ell^z(h_1)\xi(z) = T(z)\xi(z) = \lambda(z)\xi(z)$ we obtain that $d\pi_\ell^z(L)\xi(z) = \mu(z)\xi(z)$. From (6.6) we therefore get (compare [15])

$$h(z) = \langle\pi_\ell^z(F(L))\xi(z),\psi\rangle = \langle F(d\pi_\ell^z(L))\xi(z),\psi\rangle = F(\mu(z))\,\langle\xi(z),\psi\rangle,$$

i.e.

$$F \circ \mu(z) = \frac{h(z)}{\langle \xi(z), \psi \rangle}, \quad \forall z \in U \cap \mathbb{R}. \tag{6.7}$$

Now, clearly the right-hand side of (6.7) extends holomorphically to U, and thus $F \circ \mu$ extends to a holomorphic function on U.

However, from Thm. 2.2 and formula (5.8) we deduce that

$$\lim_{|\tau| \to \infty} ||\pi_\ell^\tau(h_1)|| = 0,$$

hence $\lambda(\tau) \to 0$ and $\mu(\tau) \to +\infty$ as $\tau \to \infty$. Thus the function μ is not constant, and so, varying the point $z_0 \in U \cap \mathbb{R}$ slightly, if necessary, we may assume that $\mu'(z_0) \neq 0$. Then μ is a local bi-holomorphism near z_0, and thus F has a holomorphic extension to a complex neighborhood of $\mu(z_0)$.

Q.E.D.

7 An example of a closed orbit whose restriction to the nilradical is non-closed

Let $\mathfrak{g}$ be the Lie algebra spanned by the basis

$$\mathcal{B} = \{R, S, T, X, Y, Z, M_1, M_2, N_1, N_2, \},$$

with non-trivial brackets given by

$$\begin{aligned}
&[T, X] = -X, \ [T, Y] = Y, \ [X, Y] = Z, \\
&[R, T] = M_1, \ [R, M_1] = M_2, \ [R, M_2] = -M_2, \\
&[S, T] = N_1, \ [S, N_1] = N_2, \ [S, N_2] = N_2.
\end{aligned}$$

Let $\alpha, \ \beta \in \mathbb{R} \setminus \{0\}$, and denote by ℓ the element of $\mathfrak{g}^*$ for which

$$\ell(Z) = 1, \ \ell(M_2) = \alpha, \ \ell(N_2) = \beta, \quad \ell(U) = 0$$

for all other elements U of the basis $\mathcal{B}$.

The stabilizer of ℓ in $\mathfrak{g}$ is the subspace

$$\mathfrak{g}(\ell) = \mathrm{span}\{T, \ Z, \ N_1 - N_2, \ M_1 + M_2\}.$$

Let

$$g(x, y, r, s, n, m) := \exp xX \exp yY \exp rR \exp sS \exp mM_2 \exp nN_2.$$

Then the coadjoint orbit Ω of ℓ is the subset

$$\Omega = \{\mathrm{Ad}^*(g(x, y, r, s, m, n)^{-1})\ell : \ x, y, r, s, m, n \in \mathbb{R}\}.$$

Denote by

$$\mathcal{B}^* := \{R^*, S^*, T^*, X^*, Y^*, Z^*, M_1^*, M_2^*, N_1^*, N_2^*, \}$$

the dual basis of $\mathcal{B}$. Then

$$\begin{aligned}
\Omega \ = \ & \{-mR^* + nS^* + (\alpha(e^{-r} - 1 + r) + \beta(e^s - 1 - s) - xy)T^* + \alpha(1 - e^{-r})M_1^* \\
+ \ & \alpha e^{-r} M_2^* + \beta(e^s - 1)N_1^* + \beta e^s N_2^* - yX^* + xY^* + Z^* : \ x, y, r, s, m, n \in \mathbb{R}\}.
\end{aligned}$$

We see that the restriction of Ω to the nilradical $\mathfrak{n} = \text{span}\{X, Y, Z, M_1, M_2, N_1, N_2\}$ is not closed, since, letting r tend to $+\infty$ and s to $-\infty$, the other parameters being fixed, one finds that the functionals

$$\alpha M_1^* + \beta N_1^* - yX^* + xY^* + Z^*$$

lie in the closure of $\Omega|_{\mathfrak{n}}$.

If $\alpha/\beta > 0$, then $\alpha(e^{-r} - 1 + r) + \beta(e^s - 1 - s) - xy$ tends to infinity, provided r tends to $+\infty$ and s tends to $-\infty$ whereas x, y stay bounded. Hence the orbit Ω is closed, whenever $\alpha/\beta > 0$.

On the other hand, Ω is not closed if $\alpha/\beta < 0$, since then the element $\alpha M_1^* + \beta N_1^*$ is contained in the closure of the orbit (take $s := \frac{\alpha+\beta+\alpha r}{\beta}$ and let r tend to $+\infty$.)

It is easy to see that the subspace $\mathfrak{p} := \text{span}\{T, Z, Y, M_1, M_2, N_1, N_2\}$ is the Vergne polarization for ℓ associated to the composition sequence

$$\mathfrak{g} = \mathfrak{g}_0 \supset \mathfrak{g}_1 \supset \ldots \supset \mathfrak{g}_{10} = \{0\},$$

where $\mathfrak{g}_j$ is spanned by the $j+1$-st to last element or the ordered basis

$$R, S, T, X, Y, Z, M_1, M_2, N_1, N_2$$

of $\mathfrak{g}$.

The root λ associated to the quotient space

$$\text{span}\{X, Y, M_1, M_2, N_1, N_2, Z\}/\text{span}\{Y, M_1, M_2, N_1, N_2, Z\}$$

is the linear functional $\nu := -T^*$. In particular ℓ does not satisfy Boidol's condition (B).

To fix the ideas, assume now for instance that $\alpha > 0,\ \beta > 0$ (which means that Ω is closed), and consider a real sequence $\{\tau_k\}_{k\in\mathbb{N}}$ such that $\lim_{k\to\infty} \tau_k = +\infty$. Then the sequence of orbits $\Omega_k := \Omega + \tau_k T^*,\ k \in \mathbb{N}$, tends to infinity in the orbit space. Indeed, otherwise, for a subsequence, also indexed by k for simplicity of notation, for every k there would exist an element

$$\begin{aligned} \ell_k &= (-m_k R^* + n_k S^* + \alpha(e^{-r_k} - 1 + r_k) + \beta(e^{s_k} - 1 - s_k) - x_k y_k + \tau_k)T^* \\ &+ \alpha(1 - e^{-r_k})M_1^* + \alpha e^{-r_k} M_2^* + \beta(e^{s_k} - 1)N_1^* + \beta e^{s_k} N_2^* - y_k X^* + x_k Y^* + Z^* \ \in \Omega_k, \end{aligned}$$

such that $\lim_{k\to\infty} \ell_k$ existed in $\mathfrak{g}^*$. Hence the sequences

$$\{x_k\}_k,\ \{y_k\}_k,\ \{m_k\}_k,\ \{n_k\}_k,\ \{\alpha e^{-r_k}\}_k,\ \{\beta e^{s_k}\}_k,\ \{\alpha r_k - \beta s_k + \tau_k\}_k$$

would converge. Since τ_k tends to $+\infty$, it followed that $\alpha r_k - \beta s_k$ would tend to $-\infty$ for $n \to \infty$. But $\{\alpha r_k\}_k$ and $\{-\beta s_k\}_k$ cannot tend to $-\infty$ for $n \to \infty$, since then the sequences $\{\alpha e^{-r_k}\}_k$ and $\{\beta e^{s_k}\}_k$ would diverge.

This contradiction shows that Ω_k tends to infinity in the orbit space as n tends to infinity. In particular, we see that in this example

$$\lim_{k\to\infty} ||\pi_{\ell+\tau_k T^*}(f)|| = 0,$$

for every $f \in L^1(G)$.

Q.E.D.

References

[1] P. Bernat et al., Représentations des groupes de Lie résolubles, *Dunod, Paris* 1972.

[2] J. Boidol, *-Regularity of exponential Lie groups, *Invent. math.* 56 (1980), 31–238.

[3] R. R. Coifman, G. Weiss, Transference methods in analysis, *CBMS Regional Conference Ser., A.M.S.* 31, (1977).

[4] J. Dixmier, Les C^*-algébres et leurs représentations, *Gauthier-Villars, Paris* 1964.

[5] P. Eymard, Algèbres A_p et convoluteurs de L^p, *Séminaire Bourbaki 1969/70, Lecture Notes in Math.* 180, *Springer, Berlin 1971*, pp. 55–77.

[6] A. Figà-Talamanca, Multipliers of p-integrable functions, *Bull. Amer. Math. Soc.* 70 (1964), 666–669.

[7] C. S. Herz, The theory of p-spaces with application to convolution operators, Trans. Amer. Math. Soc. 154 (1971), 69–82.

[8] C. S. Herz, Harmonic synthesis for subgroups, *Ann. Inst. Fourier* XXIII.3 (1973), 69–82.

[9] C. S. Herz, Une géneralisation de la notion de transformée de Fourier-Stieltjes, *Ann. Inst. Fourier* XXIV.3 (1974), 145–157.

[10] L. Hörmander, Hypoelliptic second-order differential equations, *Acta Math.* 119 (1967), 147–171.

[11] T. Kato, Perturbation theory for linear operators, *Springer, New York* 1966.

[12] M. A. Krasnoselskii, On a theorem of M. Riesz, *Dokl. Akad. Nauk* 131 (1959), 246–248.

[13] H. Leptin, Sur l'algèbre de Fourier d'un groupe localement compact, *C. R. Acad. Sci. Paris* 266 A (1968), 1180–1182.

[14] H. Leptin, J. Ludwig, Unitary representation theory of exponential Lie groups, *De Gruyter Expositions in Mathematics* 18, 1994.

[15] J. Ludwig, D. Müller, Sub-Laplacians of holomorphic L^p-type on rank one AN-groups and related solvable groups, *J. of Funct. Anal.* 170 (2000), 366–427.

[16] J. Marcinkiewicz, A. Zygmund, Quelques inégalités pour les opeŕations linèaires, *Fund. Math.* 32 (1939), 115–121.

[17] E. Nelson, W. F. Stinespring, Representations of elliptic operators in an enveloping algebra, *Amer. J. Math.* 81 (1959), 547–560.

[18] J.-P. Pier, Amenable Banach algebras, *Pitman Research Notes in Mathematics* 172, 1988.

[19] D. Poguntke, Auflösbare Liesche Gruppen mit symmetrischen L^1-Algebren, *J. für die Reine und Angew. Math.* 358 (1985), 20–42.

[20] D. Poguntke, oral communication.

[21] E. M. Stein, Singular integrals and differentiability properties of functions, *Princeton Univ. Press* 1970.

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