New proof of subelliptic estimates for Rockland operators

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Incomplete draft

Introduction Our results are more general than that of Głowacki: we drop the symmetry assumption and allow non-smooth operators of arbitrarily high order. Moreover trivial but useful changes - we require only discrete family of dilations which are not necessarily diagonal, we allow order to be a complex number and we allow operator valued kernels.

We think that more important is simplification of the proof.

The idea of the proof is as follows. We proceed by induction on the dimension of the group. The abelian case is easy. Then (as in [4]) we consider representations induced from the characters of the center. Estimate in representation induced from trivial character is the inductive assumption. This estimate extends to "nearby" representations. Homogeneity will give as the full set of representations, provided we can estimate L^2 norm by our operator. Here is the novelty of our approach - we observe that the estimate $\|v\| \leq C\|Pv\|$ is equivalent to $\|(1 + P^*P)^{-1}\| < 1$. Next, using the estimates from previous steps we prove that $(1 + P^*P)^{-1}$ is "locally" in $C^*(N)$. Then, by the theory of C^* algebras, the norm is realized in some irreducible representation. But in irreducible representation our operator has discrete spectrum, so the Rockland condition implies the estimate. Tricky part is that we must use kind of bootstrap argument — we are unable to give direct proof of various properties of $(1 + P^*P)^{-1}$ and we derive them via comparison with "good" operator R. In other words, before we prove the theorems we need to know that there is at least one operator for which the conclusions are valid. Such an operator do exists — we take (powers of) a generator of (stable) semigroup of probability measures, for which most of the theory is the same as general case, but few crucial steps are much easier.

Preliminaries

Let N be a nilpotent Lie group with a discrete family of dilations $\{D^k\}_{k\in\mathbb{Z}}$. We assume that

$$D^k(x)D^k(y) = D^k(xy),$$

and

$$\lim_{k\to-\infty} D^k x = 0$$

The homogeneous dimension q of N is defined by the formula

$$|D^k(F)| = 2^{kq}|F|$$

for all bounded measurable $F \subset N$.

A distribution T on N with values in linear operator between some finite dimensional vector spaces is said to be a kernel if T coincides with a locally finite measure outside any neighbourhood of origin (more general the convolver norm of T times any $C_c^\infty(N-\{0\})$ function is finite and ...). A kernel T is said to be a homogeneous kernel of order $r \in \mathbb{C}$ if T is homogeneous distribution of degree $-r-q$, that is satisfies

$$(f \circ D^k, T) = 2^{rk}(f, T)$$

for all $f \in C_c^\infty(N)$ and $t > 0$.

Remark. Ability to use vector valued kernel seem to be quite useful. We consider operator valued kernels just to make the theory more symmetric. On the other hand, one may consider vectors of distributions with different homogeneity on each coordinate — but we see no use of them. Also, multiplying values by operator instead of scalar can be reduced (under the technical conditions we need) to coordinates of different homogeneity.

A kernel T is said to be an almost homogeneous kernel of order $r \in \mathbb{R}$ if for every $f \in C_c^\infty(N-\{0\})$ $2^{-rk}\|f \circ D^k T\|_{M^1}$ is bounded when k goes to ∞.

We say that T is a kernel of positive order at most r if either T is a homogeneous kernel of order s, $0 < \Re(s) \le r$ or T is an almost homogeneous kernel of order s, $0 < s < r$.

A kernel is called regular if it coincides with a smooth function away from the origin.

We will also consider *truncated* kernels: T_1 is truncated kernel correspodning to a homogeneous kernel order r, if T and T_1 are equal in some neighborhood of 0 and T_1 is compactly supported. If $\Re(r) > 0$, then we may aproximate T by truncated kernels — defining T_n by the formula $(f, T_n) = 2^{r(n-1)}(f \circ D^{-n+1}, T_1)$ we have

$$\|(T_n - T) * f\| \le C2^{-\Re(r)n}\|f\|.$$

If T is regular we will assume that T_1 is obtained from T by multiplication with smooth function.

Example Let h be a positive Schwartz class function on N such that $\int h = 1$. Let

$$< T, f >= \sum_{k=-\infty}^{\infty} 2^{rk}(f(0) - \int h(x)f(D^k x)dx).$$

For small enough positive r, T is a regular homogeneous kernel of order r. Moreover, T generates a semigroup of probability measures. As the Levi measure equals T on $N - \{0\}$, it has density and is infinite, so (by Janssen) the semigroup has densities in L^1.

For a unitary representation π of N on a Hilbert space H and a kernel T of order r, $\Re(r) > 0$, the operator $\pi(T)$ is defined on the space $C^\infty(\pi)$ of smooth vectors for π by

$$(g, \pi(T)f) = (\phi_{f,g}, T)$$

where $\phi_{f,g}(x) = (g, \pi(x)f)$. Equivalent definition is:

$$\pi(T)f = T * \psi_f(e)$$

where $\psi_f(x) = \pi(x)f$.

As a special case of representations we get images in quotient of N. If the divisor is a homogeneous subgroup (that is invariant under D), then on quotient is well defind dilation, and the image of a homogeneous kernel is a homogeneous kernel of the same order (and similarly for almost homogeneous kernels).

(1.1). **Lemma**. *If B is compactly supported distribution, A and AB belong to $C^*(N)$, then for any unitary representation π of N*

$$\pi(A)\pi(B) = \pi(AB)$$

on $C^\infty(\pi)$.

Main theorems

(1.2). **Theorem**. *Let P be a homogeneous kernel of order r, $\Re(r) > 0$. The following conditions are equivalent:*

a) There exists homogeneous kernel S of order s, $\Re(s) = \Re(r)$ such that for every nontrivial irreducible unitary representation π of N, the operator $\overline{\pi(P)}$ is injective on the domain of $\overline{\pi(S)}$.

b) If T is a kernel of positive order at most $\Re(r)$, there exists a constant C such that

$$\forall f \in C_c^\infty(N) \qquad (\|Tf\| \le C(\|Pf\| + \|f\|)).$$

c)

$$\forall_{t>0} e^{-tP^*P} \in C^*(N)$$

d)

$$\forall_{t>0} (1 + tP^*P)^{-1} \in C^*(N).$$

If T additionaly is a regular kernel, then each of the above is equivalent to following:

e) For every nontrivial irreducible unitary representation π of N, the operator $\overline{\pi(P)}$ is injective on the space $C^\infty(\pi)$.

f) For every integer $m > 0$, and every kernel T of positive order at most $2m\Re(r)$ there exists a constant C such that

$$\forall f \in C_c^\infty(N) \qquad (\|Tf\| \le C(\|(P * P)^m f\| + \|f\|)).$$

If T is regular and takes values in square matrices, then a–f is equvalent to g:

g) For every integer $m > 0$, and every kernel T of positive order at most $m\Re(r)$ there exists a constant C such that

$$\forall f \in C_c^\infty(N) \qquad (\|Tf\| \le C(\|P^m f\| + \|f\|)).$$

If one of the equivalent conditions above is satisfied and P_1 is truncated kernel corresponding to T, then $P_1^* P_1$ is essentialy selfadjoint on $C_c^\infty(N)$.

Remark The conclusion of (1.2) means that two operators satisfying assumption of (1.2) have equal domain, so the domain in assumption of (1.2) can be chosen in canonical way.

(1.3). **Theorem**. *If T is positive definite, regular homogeneous kernel of order $r > 0$ and satisfies one of the equvalent conditions of (1.2) , then:*

a) For every unitary representation π of N the operator $\pi(T)$ is essentialy selfadjoint on $C^\infty(\pi)$, moreover T is essentialy selfadjoint on $C_c^\infty(N)$

b) For any complex s, $\Re(s) > -q/r$ the operator T^s corresponds to regular kernel of order sr and if $\Re(s) > 0$, T^s satisfies conditions of (1.2)

c) The semigroup e^{-tT} generated by T consists of smooth functions and satisfies

$$|\partial^\alpha e^{-tT}(x)| \le C_\alpha (1 + |x|)^{-q-r}.$$

(1.4). **Theorem**. *If T is regular homogeneous kernel of order r, $\Re(r) > 0$, and T satisfies one of the equvalent conditions of (1.2) , then the polar decomposition of T consists of regular kernels, that is, there exists a positive definite, regular homogeneous kernel A of order $\Re(r)$ (which satisfies equvalent conditions of (1.2)) and a regular homogeneous kernel U order $\Im(r)$ such that*

$$T = UA$$

and U gives (injective) isometry (also A is injective).

In the abelian case, the proof is easy: Fourier transform $\hat{P}$ of P is a continuous function, Rockland condition states simply that $|\hat{P}(z)| = \det(\hat{P}(z)^* \hat{P}(z))^{1/2} \neq 0$ for $z \neq 0$ so $|\hat{P}|$ (by homogeneity) majorises every continuous homogeneous function of the same order — which is the conclusion of (1.2) . Condition $e^{-tP^*P} \in C^*(N)$ is equivalent to $e^{-t|\hat{P}(z)|^2} \to 0$ when $z \to \infty$, which, thanks to homogeneity, is equivalent to $\hat{P}(z) \neq 0$ for $z \neq 0$.

Denote the center of N by V, and choose a linear complement $\tilde{N}$ to V invariant under the action of dilations.

For a functional $z \in V^*$ we shall denote by π_z the unitary representation of N induced by the character

$$v \to e^{i(v,z)}$$

from the center V of N.

The next three lemmas can be proved following the proof of (3.2) and (3.19) in [4]. One should note that regularity of kernels in [4] is only used to prove that truncated kernels of order 0 are bounded on L^2 - however the inductive argument in the proof of (3.19) works also for truncated kernels of order 0 (it *does not* work without truncation). Also the kernels $T_I = X^I T$ are no longer homogeneous, but only almost homogeneous. Fact that we admit complex orders and discrete dilations only requires that we change the notation.

(1.5). **Lemma**. *Let P be a homogeneous kernel of order r, $\Re(r) > 0$ such that for every kernel T of positive order at most $\Re(r)$ there exists a constant C such that*

$$\forall f \in C_c^\infty(\tilde{N}) \qquad (\|\pi_0(T)f\| \le C(\|\pi_0(P)f\| + \|f\|)).$$

For all $t > 0$ and for every kernel T of positive order at most $\Re(r)$ there exists another constant C such that for all $|z| \le t$

$$\forall f \in C_c^\infty(\tilde{N}) \qquad (\|\pi_z(T)f\| \le C(\|\pi_z(P)f\| + \|f\|)).$$

(1.6). **Lemma**. *Let P and T be as in (1.5) . There exists homogeneous kernel R of order r supported on V such that*

$$\forall f \in C_c^\infty(N) \qquad \|Tf\| \le C(\|Pf\| + \|Rf\| + \|f\|).$$

Moreover there is l such that

$$\forall f \in C_c^\infty(N) \qquad \|Tf\| \le C(\|Pf\| + \|(I - \Delta_V)^l f\|.$$

(1.7). **Lemma**. *Let P be a homogeneous kernel of order r, $\Re(r) > 0$ which satisfy the conclusion of (1.2) , that is for every kernel T of positive order at most $\Re(r)$ there exists constant C such that*

$$\forall f \in C_c^\infty(N) \qquad (\|Tf\| \le C(\|Pf\| + \|f\|))$$

and let P_1 be the corresponding truncated kernel. Let η be a positive function such that there exists a neighbourhood K of e and a constant C satisfying

$$\sup_{z\in K}\max_{|\alpha|\le r+q}|X^\alpha\eta(xz)|\le C\eta(z).$$

Then for every $\epsilon>0$ there exists a constant C such that

$$\forall f\in C^\infty(N)\qquad (\|[P_1,\eta]f\|\le\epsilon\|\eta P_1 f\|+C\|\eta f\|)$$

$$\forall f\in C^\infty(N)\qquad (\|\eta^{-1}[P_1,\eta]\eta f\|\le\epsilon\|\eta P_1 f\|+C\|\eta f\|).$$

Moreover, $P_1^*P_1$ is essentially selfadjoint on $C_c^\infty(N)$.

Fix a compact set $F\subset V$ such that $0\notin F$ and $V-0=\bigcup_k D^k(F)$.

(1.8). **Lemma**.

$$\exists C\forall z\in F, f\in C_c^\infty(\tilde N)(\|f\|\le C\|\pi_z(P)f\|)$$

Let as see that (1.5) and (1.8) easily imply (1.2) . Indeed (1.8) and (1.5) imply

$$\forall z\in F\forall f\in C_c^\infty(\tilde N)\qquad (\|\pi_z(T)f\|\le C(\|\pi_z(P)f\|+\|f\|)\le C_1\|\pi_z(P)f\|)$$

Using dilations, we have

$$\forall z\in\bigcup_{k\ge1}D^k(F)\forall f\in C_c^\infty(\tilde N)\qquad (\|\pi_z(T)f\|\le C_1\|\pi_z(P)f\|)$$

which, together with (1.5) gives the estimate. So it remains to prove (1.8) .

(1.8) is an immediate consequence of

(1.9). **Lemma**.

$$\sup_{z\in F}\|(I+\pi_z(P)^*\pi_z(P))^{-1}\|<1.$$

For $f\in L^1(N)$ define

$$\|f\|_{\mathrm{Alg}}=\sup_{z\in F}\|\pi_z(f)\|$$

and let Alg be the completition of $L^1(N)$ with respect to $\|\cdot\|_{\mathrm{Alg}}$. Of course Alg is a homomorphic image of $C^*(N)$ so that (non-degenerate) representations of Alg are naturally identified with representations of N. This also gives us a way to associate elements of Alg with distributions on N.

(1.10). **Lemma**. $\exists f \in \mathrm{Alg} \forall \delta$

$$\delta(f) = (1 + \delta(P)^* \delta(P))^{-1}$$

Taking (1.10) as granted we easily prove (1.9) : by the theory of C^*-algebras (see [2] Lemma 3.3.6) there exists an irreducible representation δ of Alg(hence N) such that

$$\|f\|_{\mathrm{Alg}} = \|\delta(f)\|.$$

By Kirillov theory $\delta(f)$ is compact, hence being positive either has norm smaller then 1 (which is what we want) or has eigenvector with eigenvalue 1. But since

$$\delta(f) = (1 + \delta(P)^* \delta(P))^{-1}$$

such an eigenvector must lie in the kernel of $\delta(P)$. As δ is nontrivial this contradicts the Rockland condition. So it remains to prove (1.10) .

(1.11). **Lemma**. *Let R, S be closed densely defined operators on a Hilbert space. Put $A = R^* R$, $B = S^* S$. Then*

$$\|Re^{-tA}\| \leq t^{-1/2},$$

$$\|e^{-A} - e^{-B}\| \leq 4\|R - S\|.$$

By the spectral theorem $\|Ae^{-tA}\| \leq t^{-1}$. For v in $\mathrm{dom}(A)$

$$\|Rv\|^2 = (Rv, Rv) = (R^* Rv, v) = (Av, v) \leq \|Av\|\|v\|$$

and we have

$$\|Re^{-tA}v\|^2 \leq \|Ae^{-tA}\|\|e^{-tA}\|\|v\|^2 \leq t^{-1}\|v\|^2$$

If $\|R - S\|$ is finite $\mathrm{dom}(R) = \mathrm{dom}(S)$. Thanks to this, our operators give at least well defined bilinear forms on $\mathrm{dom}(R) \times \mathrm{dom}(R)$ which provides dense common domain for calculations. We have

$$A - B = R^* R - S^* S = R^*(R - S) + (R^* - S^*)S.$$

so

$$\begin{aligned} e^{-tA}(A-B)e^{-(1-t)B} &= e^{-tA}(R^*(R-S)+(R^*-S^*)S)e^{-(1-t)B} \\ &= (Re^{-tA})^*(R-S)e^{-(1-t)B} + e^{-tA}(R-S)^*Se^{-(1-t)B} \end{aligned}$$

and

$$\begin{aligned} \|e^{-tA}(A-B)e^{-(1-t)B}\| &\le \|(Re^{-tA})^*(R-S)e^{-(1-t)B}\| + \|e^{-tA}(R-S)^*Se^{-(1-t)B}\| \\ &\le \|Re^{-tA}\|\|R-S\|\|e^{-(1-t)B}\| + \|e^{-tA}\|\|R-S\|\|Se^{-(1-t)B}\| \\ &\le (t^{-1/2}+(1-t)^{-1/2})\|R-S\|. \end{aligned}$$

Hence, using perturbation formula

$$\begin{aligned} \|e^{-A}-e^{-B}\| &= \|-\int_0^1 e^{-tA}(A-B)e^{-(1-t)B}dt\| \\ &\le \int_0^1 \|e^{-tA}(A-B)e^{-(1-t)B}\|dt \\ &\le \int_0^1 (t^{-1/2}+(1-t)^{-1/2})\|R-S\|dt = 4\|R-S\|. \end{aligned}$$

(1.12). **Lemma**. *If R is a regular kernel generating a probabilistic semigroup, and R_1 is corresponding truncated kernel $t, N > 0$, then*

$$\int e^{tR_1}(x)^2|x|^N < \infty$$

$$\int |R_1e^{tR_1}(x)|^2|x|^N < \infty$$

with uniform bound when N and t stay in a bounded set and t is bounded away from 0.

(1.13). **Lemma**.

$$\forall t > 0(e^{-tP_n^*P_n} \in C^*(N))$$

and

$$(P_n^*P_n+1)^{-1} \in C^*(N).$$

P. The first claim implies the second, so we will prove the first. We choose η to be polynomially growing, satisfy assumptions of (1.7) , and such that $L^2(\eta) \subset L^1$ – for example $(1+|x|^m)$ with m large enough. We have

$$|(P_n^* P_n f, \eta^2 f) - \|\eta P_n f\|^2| = |(P_n f, P_n \eta^2 f) - \|\eta P_n f\|^2|$$

$$= |(P_n f, [P_n, \eta]\eta f + \eta [P_n, \eta] f)|$$

$$\leq \|\eta P_n f\|(\|[P_n, \eta] f\| + \|\eta^{-1}[P_n, \eta]\eta f\|)$$

Hence, by (1.7)

$$\Re(P_n^* P_n f, \eta^2 f) \geq 1/2 \|\eta P_n e^{-tP_n^* P_n} f\|^2 - C\|\eta e^{-tP_n^* P_n} f\|^2,$$

$$|\Im(P_n^* P_n f, \eta^2 f)| \leq 1/2 \|\eta P_n e^{-tP_n^* P_n} f\|^2 - C\|\eta e^{-tP_n^* P_n} f\|^2,$$

and

$$\Re(P_n^* P_n f, \eta^2 f) + 2C\|\eta e^{-tP_n^* P_n} f\|^2 \geq |\Im(P_n^* P_n f, \eta^2 f)|.$$

Next, if $|\Im(z)| \leq \Re(z)$,

$$\partial_t \|\eta e^{-tzP_n^* P_n} f\|^2 = -2\Re(z\eta P_n^* P_n e^{-tzP_n^* P_n} f, \eta e^{-tzP_n^* P_n} f)$$

$$\leq \sqrt{2}|z|(|\Im(\eta P_n^* P_n e^{-tzP_n^* P_n} f, \eta e^{-tzP_n^* P_n} f)| - \Re(\eta P_n^* P_n e^{-tzP_n^* P_n} f, \eta e^{-tzP_n^* P_n} f)$$

$$\leq C' \|\eta e^{-tzP_n^* P_n} f\|^2$$

Consequently, $e^{-zP_n^* P_n}$ is a holomorphic family of bounded operators on $L^2(\eta)$. That means $\forall M > 0 \exists C_M \forall t < M$

$$\|\eta e^{-tP_n^* P_n} f\| \leq C_M \|\eta f\|$$

$$\|\eta P_n^* P_n e^{-tP_n^* P_n} f\| \leq \frac{1}{t} C_M \|\eta f\|.$$

Next

$$e^{-tP_n^* P_n} = e^{tR_1} + \int_0^t e^{-sP_n^* P_n}(P_n^* P_n - R_1) e^{(t-s)R_1} ds$$

$$= e^{tR_1} + \int\limits_{\epsilon}^{t-\epsilon} + \int\limits_{0}^{\epsilon} + \int\limits_{t-\epsilon}^{t} = e^{tR_1} + I_{1,\epsilon} + I_{2,\epsilon} + I_{3,\epsilon}$$

Using similar argument as in (1.11) we see that $\|I_{2,\epsilon}\|_{L^2,L^2}$ and $\|I_{3,\epsilon}\|_{L^2,L^2}$ go to 0 when ϵ goes to 0. Note that $\forall \epsilon > 0$

$$\| \int\limits_{\epsilon}^{t-\epsilon} e^{-sP_n^*P_n}(P_n^*P_n - R_1)e^{(t-s)R_1} ds\|_{L^2(\eta)}$$

$$\leq t \sup_{\epsilon \leq s \leq t-\epsilon} \left\| e^{-sP_n^*P_n}(P_n^*P_n - R_1)e^{(t-s-\epsilon/2)R_1} \right\|_{L^2(\eta),L^2(\eta)} \|e^{(\epsilon/2)R_1}\|_{L^2(\eta)} < \infty$$

so $I_{1,\epsilon} \in L^1$. Also $e^{tR_1} \in L^1$ so $e^{-tP_n^*P_n}$ is a limit in operator norm of L^1 functions so it belongs to $C^*(N)$.

(1.14). **Lemma**.

$$\forall t > 0 (e^{-tP^*P} \in C^*(N))$$

P. Since $\|P - P_n\|$ tends to 0 when n goes to infinity, (1.11) implies that e^{-tP^*P} is the norm limit of $e^{-tP_n^*P_n}$. Hence, our claim follows from (1.13) .

(1.15). **Lemma**.

$$\forall \rho \in Irr(N)(\rho(e^{-P_n^*P_n}) = e^{-\rho(P_n)^*\rho(P_n)}$$

Let R be a symmetric regular ... By (1.5) we have

$$C_0(\|Pf\| + \|f\|) \geq \|Rf\|$$

as $P - P_n$ is bounded

$$C_1(\|P_nf\| + \|f\|) \geq \|Rf\|$$

$$C_1^2(P_n^*P_n + 1) \geq R^2$$

since is central, $0 \leq$ and ...

$$(1 + C_1^2(1 + P_n^*P_n))^{-1} \leq (R^2 + 1)^{-1}.$$

As both sides of the inequality belong to $C^*(N)$ we have

$$\rho((1+C_1^2(1+P_n^*P_n))^{-1}) \leq \rho((1+R^2)^{-1}).$$

Similarly one shows that there is C_2 so that

$$\rho((1+C_2^2(1+R^2))^{-1}) \leq \rho((1+C_1^2(1+P_n^*P_n))^{-1})$$

so there is a positive definite operator $\tilde{A}$ such that

$$\rho((1+P_n^*P_n)^{-1}) = (1+\tilde{A})^{-1}.$$

Note that $\mathrm{dom}(\tilde{A}_n^{1/2}) = \mathrm{dom}(\rho(R))$ and

$$C_2(\|\rho(R)f\| + \|f\|) \geq \|\tilde{A}_n^{1/2}\|.$$

On C_ρ^∞ we have

$$\tilde{A}e^{-t\tilde{A}}f = -\partial_t e^{-t\tilde{A}}f = -\partial_t \rho(e^{-tP_n^*P_n})f =$$

$$\rho(e^{-tP_n^*P_n}P_n^*P_n)f = \rho(e^{-tP_n^*P_n})\rho(P_n^*P_n)f =$$

$$e^{-t\tilde{A}}\rho(P_n)^*\rho(P_n)f.$$

In the last line we used the fact that on smooth vectors

$$(\rho(P_n^*)f, g) = (f, \rho(P_n)g)$$

so $\rho(P_n^*) \subset \rho(P_n)^*$. Hence we got equality

$$e^{-t\tilde{A}}\tilde{A}f = e^{-t\tilde{A}}\rho(P_n)^*\rho(P_n)f.$$

If t goes to 0 we have (on smooth f)

$$\tilde{A}f = \rho(P_n)^*\rho(P_n)f.$$

As smooth vectors are a core of $\rho(P_n)$ we have $\mathrm{dom}(\rho(P_n)) \subset \mathrm{dom}(\tilde{A}_n^{1/2})$. Since $\rho(R)$ majorises $\tilde{A}_n^{1/2}$ (hence also $\rho(P_n)$) and smooth vectors are a core of $\rho(R)$ we also have

$\mathrm{dom}(\rho(R)) \subset \mathrm{dom}(\rho(P_n))$ so $\mathrm{dom}(\tilde{A}_n^{1/2}) \subset \mathrm{dom}(\rho(P_n))$. This together imply that bilinear forms associated to $\tilde{A}_n$ and $\rho(P_n)^*\rho(P_n)$ are equal so $\tilde{A}_n = \rho(P_n)^*\rho(P_n)$.

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