Analysis of Laplacians on solvable Lie groups.

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General setup

Let G be a connected Lie group, X_j right invariant vector fields on G, which generate (as a Lie algebra) the Lie algebra of G,

$$L = -\sum X_j^2.$$

If X_j linearly span the Lie algebra of G, then L is (up to first order term) the Laplace-Beltrami operator (written in a funny way) for some Riemmanian metric on G. In general L is a hypoelliptic second order operator.

L generates convolution semigroup on all L^p, $p \geq 1$ (L^p is with respect to left-invariant Haar measure)

$$e^{-tL} f = p_t * f$$

p_t are called *heat kernel*, $p_t > 0$, $p_t \in C^\infty(G)$, $p_{t+s} = p_t * p_s$. They solve the heat equation, namely if f is in L^2, $u(x,t) = p_t * f$, then

$$-Lu = \partial_t u$$

$$\lim_{t \to 0} u(t, \cdot) = f \qquad \text{in } L^2.$$

u is the unique solution which satisfies the initial condition and has uniformly (in t) bounded L^2 norms on $G \times \{t\}$.

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The closure $\mathcal{A}$ of linear span of p_t, for $t > 0$ is a (commutative) subalgebra of L^1 called the subalgebra generated by L. $\mathcal{A}$ is a $*$-Banach algebra.
Example If L is minus Laplacian on $\mathbb{R}^n$, then $\mathcal{A}$ is just set of radial L^1 functions.
L is positive definite and self-adjoint on $L^2(G)$ with respect to left Haar measure. By the spectral theorem

$$F(L) = \int_0^\infty F(\lambda) dE(\lambda)$$

is well defined operator on $L^2(G)$.
Spectral decomposition of L and Gelfand transform on $\mathcal{A}$ are closely related. For each $\lambda \in \mathrm{sp}(L)$ there is a smooth positive definite function ϕ_λ such that mapping $u \mapsto \langle u, \phi_\lambda \rangle$ is a multiplicative functional on $\mathcal{A}$ and $L\phi_\lambda = \lambda\phi_\lambda$. If G is non-compact amenable group then $\mathrm{sp}(L) = \mathbb{R}_+$.
If $u \in \mathcal{A}$ and the Gelfand transform of u in ϕ_λ equals $F(\lambda)$, then $u * f = F(L)f$.
In general $F(L)f = u * f$ where u is a distribution — in the sequel we will identify $F(L)$ with the distribution u.
We have Plancherel formula: there is a measure μ on $\mathbb{R}_+$ such that if (the function identified with) $F(L) \in L^2$ then

$$\|F(L)\|_{L^2}^2 = \int |F(\lambda)|^2 d\mu(\lambda).$$

There is also inversion formula

$$F(L)(x) = \int \phi_\lambda(x) F(\lambda) d\mu(\lambda).$$

μ is locally finite and polynomialy growing.
Remark On $\mathbb{R}^n$ Gelfand transform is basicaly the Fourier transform

$$\widehat{F(L)}(\omega) = F(|\omega|^2).$$

We begun from the heat eqation, but we may also consider the wave equation on $G \times \mathbb{R}$

$$Lu = -\partial_t^2 u.$$

If $u(0, \cdot) = f$ and $\partial_t u(0, \cdot) = h$, then

$$u(t, \cdot) = \cos(\sqrt{L})f + \frac{\sin(\sqrt{L})}{\sqrt{L}} h.$$

Two operators on non-unimodular groups

On non-unimodular groups there is another operator Δ built from X_j and closly related to L. For Δ the natural space is L^2 with respect to right-invariant Haar measure $d_r g$. We may define Δ by the formula

$$\langle \Delta f, f\rangle_r = \sum \langle X_j f, X_j f\rangle_r$$

where r means that the scalar product is with respect to $d_r g$. $d_r g$ is related to the left Haar measure dg via the modular function m:

$$d_r g = m dg$$

Then we check that

$$\Delta = -\sum (X_j + \frac{X_j m}{m}) X_j = L - \sum \frac{X_j m}{m} X_j.$$

Mapping $f \mapsto Uf = m^{1/2} f$ is an isometry of $L^2(d_r g)$ with $L^2(dg)$. We write

$$m^{-1/2} L m^{1/2} = -\sum (m^{-1/2} X_j m^{1/2})^2 = -\sum (X_j + \frac{X_j m}{2m})^2$$

$$= L - \sum \frac{X_j m}{m} X_j - \frac{1}{4} \sum \left(\frac{X_j m}{m}\right)^2 = \Delta - \frac{1}{4} \sum \left(\frac{X_j m}{m}\right)^2.$$

Hence, U intertwines the spectral resolutions of L and $\Delta - c$ where $c = \frac{1}{4} \sum \left(\frac{X_j m}{m}\right)^2$. If $e^{-t\Delta} f = q_t * f$, then

$$p_t = m^{1/2} e^{-tc} q_t.$$

Finite propagaton speed for wave equation

There is a natural metric d (*optimal control metric*) on G associated to L. For this metric we have

$$\text{supp } \cos(t\sqrt{L}) \subset B(r).$$

We have also finite propagation speed for Δ. This is related to L by the formula

$$\cos(t\sqrt{L}) = m^{1/2} (\cos(t\sqrt{\Delta - c})).$$

(1.1). **Theorem**. *If F is holomorphic and symmetric, such that*

$$|F(z)| \le C e^{r \Im z},$$

$$\int |F(\sqrt{(\lambda)}|^2 d\mu < \infty,$$

then $u = F(\sqrt{L}) \in L^2(G)$ *and* $\text{supp } u \subset B(r)$. *Conversly, if* G *admits nontrivial homomorphizm into* $\mathbb{R}$, $u \in \overline{L^2 \cap \mathcal{A}}$, $\text{supp } u \subset B(r)$, *then* $u = F(\sqrt{L})$ *with* F *as above.*

P. By euclidean Paley-Winer theorem $\text{supp } \hat{F} \subset [-r, r]$. Now

$$F(\sqrt{L}) = \int \hat{F}(t) \cos(t\sqrt{L}) dt$$

so the first claim follows from finite propagation speed. For the converse note, that $u = F(\sqrt{L})$ with some $F \in L^2(\mu)$. Now, we take image under homomorphizm and again apply euclidean Paley-Winer.

Questions

The basic question is:

Q: How does $F(L)$ behave on L^p, $p \neq 2$.

Remark. We are interested in real-variable F so L must have real spectrum. The spectrum of Δ on L^p is complex for non-unimodular G and $p \neq 2$.

Below is an atempt to present state of art, giving more specific questions, and shortly summarizing known answers.

Q1: Is $\mathcal{A}$ symmetric (if $x = x^*$, then $\mathrm{sp}(x)$ is real).

yes if $L^1(G)$ is symmetric (for exponential G, D. Poguntke [24] gave characterization), M. Christ and D. Müller [7] show G and L with non-symmetric $\mathcal{A}$, J. Ludwig and D. Müller [19] give a large class of counter-examples

Q2: Is it true that:

$$\exists_k F \in C_c^k(\mathbb{R}_+) \Rightarrow F(L) \in L^1(G)$$

yes if $\dim(G/[G,G]) = 1$, real parts of roots are positive

yes if G is Iwasawa type, L distinguished [H], [CGHM] (as pointed out by D. Poguntke, $L^1(G)$ is usually non-symmetric)

yes if $G = \mathbb{R}^k \ltimes \mathbb{R}^m$, adjoint action is semisimple, $L = L_0 + L_1$, L_0 lives on $\mathbb{R}^k$, L_1 lives on $\mathbb{R}^m$ and is a sum of eigenvectors for adjoint action (unpublished).

no on Heisenberg group extended by hyperbolic dilations [7].

Q3: Is it true that:

$$\exists_{a,b,C,k} \ \text{supp} \ F \subset [a,b] \Rightarrow \sup \|F(tL)\|_{L^1} \leq C\|F\|_{C^k}$$

yes if G is Iwasawa type, L distinguished (implicit in [11])

yes for groups considerd in this paper

Q4:Is it true that:

$$\exists_{p\neq 2,C,k,\phi} \phi \in C_c^\infty(\mathbb{R}_+), \|F(L)\|_{L^p \to L^p} \leq C \sup \|\phi F(t\cdot)\|_{C^k}$$

completely open — there is no Calderon-Zygmund theory on exponential growth groups, newertheless, we expect some positive results.

Q5: Find smallest k if the answer for one of the previous questions is positive

the critical index at infinity which we get in this paper is the best possible.

Assume that $G = A \ltimes N$, $L = X_1^2 + L_1$, X_1 generates A, L_1 lives on N and $[X_1, L_1] = 2L_1$. We consider N as a homogeneous group with dilations $D_t(x) = \exp(ctX_1)x\exp(-ctX_1)$, $c = \log(2)/2$. Note, that

$$D_k L = 2^k L.$$

N may be considerd as a stratified nilpotent Lie group, namely, its Lie algebra (which may be identified with N via exponential map) is a stratified nilpotent Lie algebra. That means there is q (we say that N is of step q) such that

$$N = \bigoplus_{j=1}^{q} V_j \ ,$$

V_1 generates N and $[V_j, V_i] \subset V_{i+j}$ for every $1 \le j, i \le q$. Usually, one assumes that D_k is a diagonal operator, and V_j are eigenspaces, but our assumption is equivalent to saying that $D_1 = \sqrt{2}U$, where U is isometry with respect to the scalar product on V_1 associated to L.
The homogeneous dimension Q of N is defined by the formula

$$|D_{2k}(A)| = 2^{kQ}|A|$$

for any measurable $A \subset N$ (of course $Q = \mathrm{tr}_N(\mathrm{ad}(X_1)) = \sum j \dim(V_j)$).
G as a set is a product of $A \approx \mathbb{R}$ and N with multiplication given by the formula

$$(s_1, n_1)(s_2, n_2) = (s_1 + s_2, D_{-s_2/c}n_1 \cdot n_2).$$

For further reference note that the Lebesque measure is left invariant and the modular function m is given by the formula $m(s, n) = e^{Qs}$.

(1.2). **Theorem**. *Let $F_t(\lambda) = F(t\lambda)$. If $s_0 > \frac{Q+1}{2}$, $s_1 > \frac{3}{2}$,*

$$\int_{t>1} \|\phi F_t\|_{H(s_1)} \frac{dt}{t} < \infty$$

and

$$\int_{t\le 1} \|\phi F_t\|_{H(s_0)} \frac{dt}{t} < \infty$$

then $F(L)$ is bounded on L^1.

Remark. From (1.2) and trivial L^2 estimate one can easily interpolate an L^p theorem, which however does not seem to be sharp.

(1.3). **Lemma**. *Let $q, Q \in \mathbb{R}_+$, $q \le Q$, supp $f \subset [2^{k-1}, 2^{k+1}]$, $f \in H(s)$, $s \ge 0$, then there exist symmetric functions f_l, such that*

$$f = \sum f_l \qquad \text{on } \mathbb{R}_+,$$

$$\mathit{supp}\ \hat{f}_l \subset [-2^{l-k}, 2^{l-k}],$$

for $k \ge 1$,

$$\int_0^\infty |f_l(x)|^2 (x^{2q} + x^{2Q}) dx \le C 2^{-2sl} 2^{(2q+1)k} \|\delta_{2^k} f\|_{H(s)}^2$$

and for $k \le 1$,

$$\int_0^\infty |f_l(x)|^2 (x^{2q} + x^{2Q}) dx \le C 2^{-2sl} 2^{(2Q+1)k} \|\delta_{2^k} f\|_{H(s)}^2$$

P. We may assume that $k = 0$ (otherwise we replace f by $\delta_k f$). Next, we extend f symmetricaly to negative halfline. We choose $\phi \in C_c^\infty(\mathbb{R})$ such that $\phi(x) = 1$ for $|x| < \frac{1}{2}$ and $\phi(x) = 0$ for $|x| \ge 1$. We prescribe the Fourier transform of f_l:

$$\hat{f}_0 = \phi \hat{f},$$

and for $l > 0$

$$\hat{f}_l = \left(\phi(2^{-l}x) - \phi(2^{-l+1}x)\right) \hat{f}.$$

By definition we have correct support of $\hat{f}_l$ and

$$\|f_l\|_{L^2} \le C 2^{-sl} \|f\|_{H(s)}.$$

We write

$$f_l = (\psi_l - \psi_{l-1}) * f$$

where $\hat{\psi} = \phi$ and $\psi_l(x) = 2^l \psi(2^l x)$. Since supp $f \subset [-2, 2]$ and

$$|\psi_l(x) \le C 2^l (1 + 2^l |x|)^{-Q-s-1}$$

so

$$\int_{|x|>4} |f_l|^2 (1 + |x|)^{2Q} \le \left(\int |f|\right)^2 C \int_{|x|>4} 2^{2l} (2^l (|x| - 2)^{-2Q-2s-2} (1 + |x|)^{2Q}$$

$$\leq C\|f\|^2_{H(s)} I$$

Now $(|x|-2) \geq 1$ so

$$(2^l(|x|-2))^{-2s} \leq 2^{-2sl}.$$

Moreover

$$(2^l(|x|-2))^{-2Q}(1+|x|)^{2Q} \leq C.$$

Finally

$$I \leq C2^{-2sl} \int\limits_{|x|>4} 2^{2l}2^{-2l}(|x|-1)^{-2} \leq C2^{-2sl}$$

Homogeneous estimates

We fix a homogeneous norm on N, that is a function $|\cdot|$ on N such that

$$|x| \geq 0, \qquad |x| = 0 \iff x = 0 \qquad |D_{2k}(x)| = 2^k|x|.$$

We put $w(s,n) = |n|^Q$. We will consider w both as function on N and as a function on G (independent on the coordinate from A).

In this section we consider functions on N, so we will simply write L instead of L_1.

On N since L is homogeneous μ is homogeneous too:

$$\|F(\sqrt{L})\|^2_{L^2} = c\int\limits_0^\infty |F(x)|^2 x^{Q-1} dx.$$

(1.4). **Theorem**. *If supp $f \subset [1/2, 2]$, $\varepsilon > 0$, then*

$$\|f(\sqrt{L})\|_{L^1} \leq C\|f(\sqrt{L})\|_{L^2(1+|x|^{Q+\varepsilon})} \leq C'\|f\|_{H(\frac{Q}{2}+\varepsilon)}.$$

P. Apply (1.3) and Plancherel on N.

(1.5). **Lemma**. *On N*

$$\|\sum D_k f_k\|^2_{L^2(w)} \leq C_\varepsilon \sum \|f_k\|^2_{L^2(1+|x|^{Q+\varepsilon})}$$

P. $D_k f_k$ are almost orthogonal in $L^2(w)$.

Let $J = x\partial_x$ be generator of dilations on $\mathbb{R}^1$.

(1.6). **Lemma**. *If* $s > \frac{Q}{2}$, $n > \frac{Q}{2}$, $f(t) \to 0$ *when* $t \to \infty$, *then*

$$\|f(L)\|^2_{L^2(w)} \le C \int \|\phi f(t\cdot)\|^2_{H(s)} \frac{dt}{t},$$

$$\|f(L)\|^2_{L^2(w)} \le C \int \left(|J^n f|^2(t) + |Jf|^2(t)\right) \frac{dt}{t}.$$

P. To prove the first inequality, let ϕ be a C^∞ function on $\mathbb{R}_+$ such that $\sum \phi(2^k t) = 1$ for $t > 0$. Put $f_k(t) = f(2^k t)\phi(t)$. We have

$$f(L) = \sum f_k(2^k L) = \sum D_k f_k(L)$$

and

$$\sum \|f_k\|^2_{H(s)} \le C \int \|\phi f(t\cdot)\|^2_{H(s)} \frac{dt}{t}$$

However, one can show that

$$\|f_k(L)\|^2_{L^2(1+|x|^{Q+\varepsilon})} \le C\|f_k\|^2_{H(s)}$$

so the first inequality follows from (1.5) .
Put $c_k = f(2^{k+1}) - f(2^k)$.

$$|c_k|^2 = |\int_{2^k}^{2^{k+1}} Jf \frac{dt}{t}|^2 \le \int_{2^k}^{2^{k+1}} |Jf|^2 \frac{dt}{t} \int_{2^k}^{2^{k+1}} \frac{dt}{t} \le \log(2) \int_{2^k}^{2^{k+1}} |Jf|^2 \frac{dt}{t}$$

so

$$\sum |c_k|^2 \le \log(2) \int |Jf|^2 \frac{dt}{t}.$$

Let ψ be a C^∞ function on $\mathbb{R}_+$ such that $\psi(t) = 0$ for $t \ge 2$ and $\psi(t) = 1$ for $t \le 1$. Put $g(t) = -\sum c_k \psi(2^{-k}t)$, $h = f - g$. To see that series defining g is convergent note that for given t there is k_0 such that terms with index $k < k_0$ are equal to 0 and terms with index $k > k_0$ are equal to c_k. Also $\sum_{k=l}^m c_k = f(2^{m+1}) - f(2^l)$. Since $f(t)$ goes to 0 when t goes to ∞, the series of c_k is convergent. This argument also shows that for integral l we have $g(2^l) = f(2^l)$. Now

$$g(L) = -\sum c_k D_{-k}\psi(L)$$

and (by (1.5))

$$\|g(L)\|^2_{L^2(w)} \le C \sum |c_k|^2 \le C' \int |Jf|^2 \frac{dt}{t}.$$

Next, $h(2^k) = 0$ so

$$\int_{2^k}^{2^{k+1}} |h|^2 \frac{dt}{t} \leq \int_{2^k}^{2^{k+1}} \left(\int_{2^k}^{t} |Jh|(s) \frac{ds}{s} \right)^2 \frac{dt}{t}$$

$$\leq \log(2) \int_{2^k}^{2^{k+1}} \int_{2^k}^{t} |Jh|^2(s) \frac{ds}{s} \frac{dt}{t}$$

$$\leq \log(2)^2 \int_{2^k}^{2^{k+1}} |Jh|^2(t) \frac{dt}{t}$$

so

$$\int |h|^2 \frac{dt}{t} \leq \log(2)^2 \int |Jh|^2(t) \frac{dt}{t}.$$

Now, we get the second inequality applying the first inequality to h.

(1.7). **Lemma**. *If* $N = \mathbb{R}^{2n}$, $L = \sum_{j=1}^{2n} \partial_j^2$, *then*

$$\|f(L)\|^2_{L^2(w)} \geq c \int_0^\infty \left(|J^n f|^2 + |Jf|^2\right) \frac{dt}{t}$$

P. One easily checks that left side has form

$$\int |\sum_{k=1}^{n} c_k J^k f|^2 \frac{dt}{t}.$$

For $n = 1$ this gives the claim. Let L_n be laplacian on $\mathbb{R}^{2n}$. Strightforward calculation shows that

$$\|f(L_n)\|^2_{L^2(w)} \leq C_n \|f(L_{n+1})\|^2_{L^2(w)}.$$

Now we proceed by induction on n: we get the estimate for lower order terms from the inductive assumption.

AN groups

The distance naturally associated to L (optimal control metric) for our groups is given by the formula

$$d(s, n) = \operatorname{arccosh}\left(\frac{e^{-2s} + 1 + |n|^2}{e^{-s}}\right)$$

provided that $|n|$ is the optimal control distance between n and e associated to L_1 on N. Put $B_r = \{x \in G : d(x, e) < r\}$.

(1.8). **Lemma.**

$$\int_{B_r} (1+w)^{-1} \le r^2$$

For Iwasawa AN groups we know the Plancherel measure of L — there is a μ such that

$$\|f(L)\|^2 = \int |f(\lambda)|^2 \, d\mu(\lambda)$$

where $d\mu = h(\lambda)dx$, h is bounded by a polynomial and $h(\lambda) \approx \lambda^{\frac{1}{2}}$ for small λ (cf. for example [11]). What we need is the following estimate

$$\|f(\sqrt{L})\|^2 \le C \int |f(\lambda)|^2 (\lambda^2 + \lambda^Q) d\lambda.$$

Main property of Iwasawa AN group is that Δ is invariant under a large group – in the rank 1 case isometries are transitive on spheres. This implies that functions of Δ are radial, and that for L we have

$$F(L)(x) = m^{1/2}\nu(x)$$

where ν is radial.

(1.9). **Lemma.**

$$\int_{B_r} |f(L)|^2 w \le Cr \int_{B_r} |f(L)|^2$$

P. $\psi = f(L)$ has the following form

$$\psi(s,n) = e^{\frac{-Qs}{2}} \nu(d((s,n),e))$$

where ν is function on $\mathbb{R}_+$. On most of the annulus $r-1 < d((s,n),e) < r$ we approximatly have

$$|n| \approx e^{\frac{r-s}{2}}$$

(more precisely $|n|$ is a function of r and s, if $d((s,n),e) < r$ then $|n| < e^{\frac{r-s}{2}}$, and if $|s| < r-1$, $d((s,n),e) = r$, then $|n| > ce^{\frac{r-s}{2}}$), so

$$\int_{r-1<d((s,n),e)<r} \psi^2 \approx \int_{r-1}^{r} \nu^2 \int_{-r}^{r} e^{Qs} e^{\frac{Q(r-s)}{2}} \approx e^{Qr} \int_{r-1}^{r} \nu^2$$

and

$$\int_{r-1<d((s,n),e)<r} \psi^2 w \approx \int_{r-1}^{r} \nu^2 \int_{-r}^{r} e^{Qs} e^{Q(r-s)} \approx re^{Qr} \int_{r-1}^{r} \nu^2$$

Let $\mathcal{A}_1$ be the subalgebra of $L^1(N)$ generated by L_1.

(1.10). **Lemma**. *If $f(\lambda) = \sum c_k e^{-t_k \lambda}$, $t_k > 0$, then $f(L) \in L^1(ds, \mathcal{A}_1)$, moreover the Gelfand transform of $f(L)$ along N is independent of N*

P. First consider $N = \mathbb{R}$. Mapping $(s, n) \mapsto (s, -n)$ is an automorphism of G preserving L, so p_t is symmetric in n. Put

$$h_t(s, \lambda^2) = \int p_t(s, n) e^{i\lambda n} dn.$$

$h_t(s, \lambda)$ is defined for $\lambda \geq 0$. Since p_t is holomorphic semigroup on L^1

$$\partial_t h_t(s, \lambda) = \partial_s^2 h_t(s, \lambda) - e^{-2s} \lambda h_t(s, \lambda),$$

$$\int \sup_\lambda |h_t(s, \lambda)| \, ds = 1$$

and

$$\lim_{t \to 0_+} h_t(s, \lambda) = \delta(s).$$

We return now to general N. It is easy to see that putting

$$T(t)(s, n) = h(s, L_1)(n)$$

we get continuous family of contractions on $L^2(G)$. Moreover

$$\partial_t T(t) = (\partial_s^2 - e^{-2s} L_1) T(t) = L T(t),$$

$$\lim_{t \to 0_+} T(t) = I$$

hence $T(t) = e^{-tL}$. This gives the conclusion for p_t, that is for $f(\lambda) = e^{-t\lambda}$, and than we get the claim by linearity.

(1.11). **Lemma**. *If supp $\hat{f} \subset [-r, r]$, then for $r > 1$, and even Q,*

$$\|f(\sqrt{L})\|_{L^1} \leq C r^{3/2} \left(\int_0^\infty |f(x)|^2 (x^2 + x^{Q+2}) dx \right)^{1/2}$$

and for $r \leq 1$,

$$\|f(\sqrt{L})\|_{L^1} \leq C r^{(Q+1)/2} \left(\int_0^\infty |f(x)|^2 (x^2 + x^{Q}) dx \right)^{1/2}$$

P. If $r \leq 1$, then

$$\|f(\sqrt{L})\|_{L^1} \leq \left(\int\limits_{|x|<r} 1 \right)^{1/2} \|f(\sqrt{L})\|_{L^2}$$
$$\leq Cr^{(Q+1)/2} \left(\int\limits_0^\infty |f(x)|^2 (x^2 + x^Q) dx \right)^{1/2}.$$

If $r > 1$, then Q is even. Next, let $\tilde{L}$ be the laplacian on Iwasawa type AN with $N = \mathbb{R}^{Q+2}$. We have

$$\|f(\sqrt{L})\|_{L^1} \leq \left(\int\limits_{|x|<r} \frac{1}{1+w} \right)^{1/2} \|f(\sqrt{L})\|_{L^2(1+w)}$$

Now by (1.6) , (1.7) , (1.10) and (1.9)

$$\|f(\sqrt{L})\|^2_{L^2(1+w)} \leq C_1 \|f(\sqrt{\tilde{L}})\|^2_{L^2(1+w)}$$
$$\leq C_2 r \|f(\sqrt{\tilde{L}})\|^2_{L^2}$$

so

$$\|f(\sqrt{L})\|_{L^1} \leq Cr^{3/2} \left(\int\limits_0^\infty |f(x)|^2 (x^2 + x^{Q+2}) dx \right)^{1/2}$$

Proof of (1.2) : We split F. Let ψ be C_c^∞ function such that supp $\psi \subset [1/2, 2]$ and for all $x > 0$ $\sum \psi(2^k x) = 1$. We write $F(x) = \sum F_k(x)$, where $F_k(x) = m(x)\psi(2^{-k}x)$. It is enough to prove that for $k \leq 0$ and $s > 3/2$ we have

$$\|F_k(\sqrt{L})\|_{L^1} \leq C \|\delta_{2^k} F_k\|_{H(s)}$$

and that for $k \geq 0$ and $s > (Q+1)/2$ we have

$$\|F_k(\sqrt{L})\|_{L^1} \leq C \|\delta_{2^k} F_k\|_{H(s)}.$$

Fix $k \leq 0$. We split $f = F_k$ using (1.3) (with Q replaced by even number greater then $Q+2$). By (1.11)

$$\|f_l(\sqrt{L})\|_{L^1} \leq C(2^{l-k})^{3/2} 2^{-sl} 2^{3k/2} \|\delta_{2^k} F_k\|_{H(s)}$$
$$= C 2^{(\frac{3}{2}-s)l} \|\delta_{2^k} F_k\|_{H(s)}$$

so

$$\|f(\sqrt{L})\|_{L^1} \leq \sum \|f_l(\sqrt{L})\|_{L^1} \leq C' \|\delta_{2^k} F_k\|_{H(s)}.$$

Case $k \geq 0$ is similar.

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