MULTIPLIERS AND SINGULAR INTEGRALS ON EXPONENTIAL GROWTH GROUPS

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ABSTRACT. We propose a simple abstract version of Calderón–Zygmund theory, which is applicable to spaces with exponential volume growth, and then show that important specific operators can be treated within this framework.

1. ABSTRACT CALDERÓN–ZYGMUND THEORY

Throughout we use the usual convention that C stands for a positive constant, usually a large positive constant, whose precise value varies from occurrence to occurrence. However, to avoid writing such curiosities as "$C + C + 3C \le C$" we sometimes use $C', C'', \ldots$ instead of C. The precise values of $C', C'', \ldots$ also change from occurrence to occurrence.

Definition 1.1. We say that the space M with metric d and Borel measure μ has the *Calderón–Zygmund property* if there exists a constant C such that for every f in L^1 and for every $\lambda > C\frac{\|f\|_{L^1}}{\mu(M)}$ ($\lambda > 0$ if $\mu(M) = \infty$) we have a decomposition $f = \sum f_i + g$, such that there exist sets Q_i, numbers r_i, and points x_i satisfying:

- $f_i = 0$ outside Q_i,
- $\int f_i \, d\mu = 0$,
- $Q_i \subset B(x_i, Cr_i)$,
- $\sum \mu(Q_i^*) \le C\frac{\|f\|_{L^1}}{\lambda}$, where $Q_i^* = \{x : d(x, Q_i) < r_i\}$,

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- $\sum \|f_i\|_{L^1} \leq C\|f\|_{L^1}$,
- $|g| \leq C\lambda$.

Since $g = f - \sum f_i$, we have $\|g\|_{L^1} \leq C'\|f\|_{L^1}$, hence $\|g\|_{L^2}^2 \leq C''\lambda\|f\|_{L^1}$.

If $K(x,y)$ is any measurable kernel defined on $M \times M$, then let K also denote the associated integral operator:

$$(Kf)(x) = \int_M K(x,y)f(y)\,dy\ .$$

Theorem 1.2. *If M has the Calderón–Zygmund property, if $T = \sum_{n\in\mathbb{Z}} K_n$ is bounded on L^2, and if for constants C, $0 < c < 1$, $a > 0$, $b > 0$*

$$\int |K_n(x,y)|(1 + c^n d(x,y))^a\,dx \leq C\ ,$$

$$\int |K_n(x,y) - K_n(x,z)|\,dx \leq C(c^n d(y,z))^b\ ,$$

then T is of weak type $(1,1)$ and bounded on L^p, $1 < p \leq 2$.

Remark 1.3. If $\sum K_n$ is strongly convergent on L^2, then easy extensions to our arguments show that the sum is also strongly convergent on L^p, $1 < p \leq 2$ and is convergent in measure for arguments in L^1. Similarly, almost everywhere convergence on L^2 implies almost everywhere convergence on L^p, $1 \leq p \leq 2$.

Remark 1.4. We find the given formulation very convenient. However, the assumption about K_n may be replaced by the weaker condition

$$\int_{d(x,y),d(x,z)>\varepsilon d(y,x)} |K(x,y) - K(x,z)|d\mu(x) \leq C_\varepsilon$$

with $\varepsilon > 0$ small enough.

Lemma 1.5. *Let f_i, Q_i, r_i, x_i, and Q_i^* be as in 1.1. Then there exists C such that for all i*

$$\sum_{n:c^n r_i \geq 1} \int_{(Q_i^*)^c} |K_n f_i|(x)\,dx \leq C\|f_i\|_{L^1}\ .$$

Proof.

$$\int_{(Q_i^*)^c} |K_n f_i|(x)\,dx$$
$$\le \|f_i\|_{L^1} \sup_{y\in Q_i} \int_{(Q_i^*)^c} |K_n(x,y)|\,dx \le \|f_i\|_{L^1} \sup_{y} \int_{d(x,y)\ge r_i} |K_n(x,y)|\,dx$$
$$\le (c^n r_i)^{-a} \|f_i\|_{L^1} \sup_{y} \int_{d(x,y)\ge r_i} |K_n(x,y)|(1+c^n d(x,y))^a\,dx$$
$$\le C(c^n r_i)^{-a} \|f_i\|_{L^1} \ .$$

Hence,

$$\sum_{n:c^n r_i\ge 1} \int_{(Q_i^*)^c} |K_n f_i|(x)\,dx \le C\|f_i\|_{L^1} \sum_{n:c^n r_i\ge 1} (c^n r_i)^{-a} \le C'\|f_i\|_{L^1} \ . \quad \square$$

Lemma 1.6. *Let f_i, Q_i, r_i, x_i, and Q_i^* be as in 1.1. Then there exists C such that for all i*

$$\sum_{n:c^n r_i<1} \int |K_n f_i|(x)\,dx \le C\|f_i\|_{L^1} \ .$$

Proof. Since $\int f_i(y)dy = 0$,

$$\begin{aligned}
\int |K_n f_i|(x)\,dx &= \int \left| \int_{Q_i} K_n(x,y) f_i(y) dy \right| dx \\
&= \int \left| \int_{Q_i} (K_n(x,y) - K_n(x,x_i)) f_i(y) dy \right| dx \\
&\le \int \int_{Q_i} |K_n(x,y) - K_n(x,x_i)| |f_i(y)| dy\,dx \\
&\le \|f_i\|_{L^1} \sup_{y\in Q_i} \int |K_n(x,y) - K_n(x,x_i)|\,dx \\
&\le C\|f_i\|_{L^1} \sup_{y\in Q_i} (c^n d(y,x_i))^b \le C'(c^n r_i)^b \|f_i\|_{L^1} \ .
\end{aligned}$$

Hence,

$$\sum_{n:c^n r_i<1} \int |K_n f_i|(x)\,dx \le \sum_{n:c^n r_i<1} C'(c^n r_i)^b \|f_i\|_{L^1} \le C''\|f_i\|_{L^1} \ . \quad \square$$

Proof of 1.2. By the Marcinkiewicz interpolation theorem it is enough to show that T is of weak type $(1,1)$. Fix $\lambda > 0$. If $\lambda \le C\frac{\|f\|_{L^1}}{\mu(M)}$, then

$$\mu(\{x : |Tf| > \lambda\}) \le \mu(M) \le C\frac{\|f\|_{L^1}}{\lambda}$$

and we are done. So assume that $\lambda > C\frac{\|f\|_{L^1}}{\mu(M)}$ and fix a Calderón–Zygmund decomposition of f. Put $E = \{x : \sum_{n,i} |K_n f_i| > \frac{\lambda}{2}\}$, $E_1 = \bigcup_i Q_i^*$. By 1.5 and 1.6,

$$\int_{E_1^c} \sum_{n,i} |K_n f_i| \;\le\; \sum_i \int_{(Q_i^*)^c} \sum_n |K_n f_i| \;\le\; C \sum_i \|f_i\|_{L^1} \;\le\; C' \|f\|_{L^1}$$

so

$$|E - E_1| \le \frac{2C'\|f\|_{L^1}}{\lambda} \ .$$

Now

$$\begin{aligned} &|\{x : |Tf(x)| > \lambda\}| \le |\{x : |Tg(x)| > \frac{\lambda}{2}\}| + |E| \\ &\le \frac{4\|Tg\|_{L^2}^2}{\lambda^2} + |E_1| + |E - E_1| \le \frac{C''\lambda\|f\|_{L^1}}{\lambda^2} + C\frac{\|f\|_{L^1}}{\lambda} + \frac{2C'\|f\|_{L^1}}{\lambda} \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \le C'''\frac{\|f\|_{L^1}}{\lambda} \ . \qquad \square \end{aligned}$$

Remark 1.7. The above proof of 1.2 remains valid using the weaker hypotheses of sublinearity, namely

$$|(Tf)(x)| \le \sum_n |(K_n f)(x)| \ ,$$

$$|(K_n(f_1 + f_2))(x)| \le |(K_n f_1)(x)| + |(K_n f_2)(x)| \ .$$

2. Main theorems

Let T be a homogeneous tree of order $q + 1$ (so each vertex has $q + 1$ neighbours). On T we consider the natural distance d (length of the shortest path between two vertices). We fix an infinite geodesic g in T. We fix a numeration of the vertices in g (so we get a mapping $N : g \mapsto \mathbb{Z}$ such that $|N(x) - N(y)| = d(x, y)$ for $x, y \in g$).

Think of the tree as hanging down from the point at infinity defined by the ray of g where $N \to +\infty$. With that picture in mind we define the level function, $l(x)$, by the formula $l(x) = N(x') - d(x, x')$, where x' is the unique vertex of g closest to x. Let $(Af)(x) = \frac{1}{2\sqrt{q}} \sum_{d(x,y)=1} q^{(l(y)-l(x))/2} f(y)$. For a complex function f on the tree we define the gradient ∇f by the formula

$$(\nabla f)(x) = \sum_{y:d(x,y)=1} |f(y) - f(x)| \ .$$

We define the measure μ on T by the formula

$$\int f \, d\mu = \sum f(x) q^{l(x)} \ .$$

Note that A is self-adjoint on $L^2(d\mu)$. Also note that $\|A\|_{L^1(d\mu) \to L^1(d\mu)} \leq 1$, hence that $\|A\|_{L^p(d\mu) \to L^p(d\mu)} \leq 1$ for $1 \leq p \leq \infty$. Thus the L^2-spectrum of A lies in $[-1, 1]$. For any $\theta \in \mathbb{R}$ the function $q^{i\theta l(x)}$ is "almost" in $L^2(d\mu)$. More precisely $q^{i\theta l(x)} e^{-\varepsilon d(x,x_0)} \in L^2(d\mu)$ for any $\varepsilon > 0$ and $x_0 \in T$. One calculates that $A(q^{i\theta l(x)}) = \cos\theta \, q^{i\theta l(x)}$ and a slightly more involved calculation using $q^{i\theta l(x)} e^{-\varepsilon d(x,x_0)}$ shows that $\cos\theta$ is in the L^2-spectrum of A. Hence:

Remark 2.1. The L^2-spectrum of A is precisely $[-1, 1]$.

Notation 2.2. If $t > 0$ and if $H(\lambda)$ is any function defined for real λ,

$$(\mathcal{D}_t H)(\lambda) = H(t\lambda) \ .$$

Theorem 2.3. *Fix any nonzero $\phi \geq 0$ in $C_c^\infty([1/4, 2])$. Suppose that $F(\lambda) = H(1 + \lambda)$ (or $F(\lambda) = H(1 - \lambda)$) where $\operatorname{supp} H \subset [0, 2)$. If for some $s > 3/2$*

$$\sup_{t>0} \|(\mathcal{D}_t H)\phi\|_{H(s)} < \infty \ ,$$

then $F(A)$ is of weak type $(1, 1)$ and bounded on $L^p(d\mu)$, $1 < p < \infty$. Moreover $\nabla(I - A)^{-1/2}$ is of weak type $(1, 1)$ and bounded on L^p, $1 < p \leq 2$.

Let $G = \mathbb{R} \ltimes \mathbb{R}^Q$ with $s \in \mathbb{R}$ acting on $\mathbb{R}^Q$ by $n \mapsto e^{-s}n$. Multiplication is given by

$$(s_1, n_1)(s_2, n_2) = (s_1 + s_2, e^{s_2}n_1 + n_2)\ .$$

This G is the Iwasawa (or AN) group corresponding to the real rank 1 symmetric space $SO_+(Q+1, 1)/SO(Q)$. There is a distinguished Laplacian on G, which is up to a first order term (or up to conjugation) equal to the Laplace–Beltrami operator on the symmetric space. *Right-invariant* vector fields are

$$X_0 = \partial_a\ , \qquad\qquad X_i = e^a \partial_{n_i} \qquad \text{for } i = 1, \ldots, Q.$$

Then we put

$$L = -\sum_{0 \le i \le Q} X_i^2\ .$$

We consider L on $L^p(G)$ with respect to *left-invariant* Haar measure dx (which is equal to Lebesgue measure in our coordinates).

Theorem 2.4. *Fix any nonzero $\phi \ge 0$ in $C_c^\infty([1/2, 2])$. If $s_0, s_1 > \frac{3}{2}$, $s_1 > \frac{Q+1}{2}$,*

$$\sup_{t \ge 1} \|(\mathcal{D}_t F)\phi\|_{H(s_0)} < \infty\ ,$$

$$\sup_{0 < t \le 1} \|(\mathcal{D}_t F)\phi\|_{H(s_1)} < \infty\ ,$$

then $F(L)$ is of weak type $(1, 1)$ and bounded on L^p, $1 < p < \infty$. Moreover $\nabla L^{-1/2}$ is of weak type $(1, 1)$ and bounded on L^p, $1 < p \le 2$.

3. Covering lemma on the weighted tree

We defined the measure μ on T by the formula

$$\int f\, d\mu = \sum f(x) q^{l(x)}\ .$$

We will write $|R|$ as a shorthand for the measure $\mu(R)$. Say that x lies *below* y (and that y lies *above* x) if $l(x) = l(y) - d(x, y)$. Say that

$R \subset T$ is an *admissible trapezoid* iff R consists of a single point x_R, or if there is $x_R \in T$ and an integer $h > 0$, such that R consists of all x below x_R such that $h \leq l(x_R) - l(x) < 2h$. Put $h(R) = h$ in this second case and $h(R) = 1$ in the one point case. In either case one finds $|R| = h(R)q^{l(x_R)}$. We call $w(R) = q^{l(x_R)}$ the *width* of R.

For an admissible trapezoid R we define its *envelope* $\tilde{R}$ as follows: if R consists of one point, then $\tilde{R} = R$, otherwise $\tilde{R}$ consists of all x below x_R such that $h/2 \leq l(x_R) - l(x) < 4h$. Note that if $R_1 \cap R_2 \neq \emptyset$ and $w(R_1) \geq w(R_2)$, then $R_2 \subset \tilde{R}_1$. It is also easy to check that $|\tilde{R}| \leq 4|R|$. We define a maximal function M by the formula

$$(Mf)(x) = \sum_{x\in R} |R|^{-1} \int |f(y)|\, d\mu(y) = \sup_{x\in R} |R|^{-1} \sum_{y\in R} |f(y)| q^{l(y)}$$

where the sup is taken over admissible trapezoids R.

Theorem 3.1. *T with measure μ and distance d has the Calderón–Zygmund property. Furthermore M is of weak type* $(1,1)$.

Proof. Let $f \in L^1(d\mu)$. Let S_0 be the family of all admissible trapezoids R such that

$$\sum_{x\in R} |f(x)| q^{l(x)} \geq \lambda |R| \ .$$

Start by listing S_0 in some arbitrarily chosen order. Choose R_0 to be an admissible trapezoid in S_0 of largest width (possible since the width is bounded by $|R|$, and $|R| \leq \|f\|_{L^1}/\lambda$). In case of ties, choose that trapezoid of largest width which occurs earliest in the listing of S_0. Now we proceed inductively: S_{i+1} consists of all $R \in S_i$ disjoint from $R_0, \dots, R_i$ and R_{i+1} is an admissible trapezoid in S_{i+1} of largest width. Since the R_i are disjoint, and since each $R_i \in S$ we have

$$\sum_i |R_i| \leq \frac{1}{\lambda} \sum_i \sum_{x\in R_i} |f(x)| q^{l(x)} \leq \frac{\|f\|_{L^1}}{\lambda} \ .$$

Consequently

$$\sum_i |\tilde{R}_i| \le 4 \sum_i |R_i| \le \frac{4\|f\|_{L^1}}{\lambda} \ .$$

If $\sum_{x \in R} |f(x)| q^{l(x)} \ge \lambda |R|$, then by construction R intersects some R_i with width not smaller then the width of R, hence with $R \subset \tilde{R}_i$. So, putting $E = \bigcup \tilde{R}_i$, we have $Mf \le \lambda$ outside E, and $|E| \le \frac{4\|f\|_{L^1}}{\lambda}$, so the second claim is proved.

To get the f_i in the Calderón–Zygmund decomposition, we first define auxiliary sets U_i and functions h_i as follows:

$$U_i = \tilde{R}_i - \bigcup_{j<i} \tilde{R}_j \ ,$$

$h_i(x) = f(x)$ for $x \in U_i$ and $h_i(x) = 0$ for $x \notin U_i$. We claim that

$$\sum_x |h_i(x)| q^{l(x)} \le 6q\lambda |\tilde{R}_i| \ .$$

Indeed it is easy to find three admissible trapezoids P_1, P_2, P_3, such that $w(P_k) > w(R_i)$ $(k = 1, 2, 3)$, $|P_k| \le 2q|\tilde{R}_i|$, and $\tilde{R}_i \subset P_1 \cup P_2 \cup P_3$. If

$$\sum_{x \in P_k} |f(x)| q^{l(x)} \ge \lambda |P_k| \ ,$$

then there is $j < i$ such that $P_k \cap R_j \ne \emptyset$, hence $P_k \subset \tilde{R}_j$, hence h_i is zero on P_k. One way or the other $\sum_{x \in P_k} |h_i(x)| q^{l(x)} \le \lambda |P_k|$, which means that

$$\sum_x |h_i(x)| q^{l(x)} \le \sum_{k=1}^{3} \sum_{x \in P_k} |h_i(x)| q^{l(x)} \le \sum_{k=1}^{3} \lambda |P_k| \le 6q\lambda |\tilde{R}_i|$$

as claimed. We put

$$f_i = h_i - \left(\sum_x h_i(x) q^{l(x)} \right) \frac{\chi_{R_i}}{|R_i|} \ , \qquad\qquad g = f - \sum f_i \ .$$

Now we put $Q_i = \tilde{R}_i$, $r_i = h(R_i)/4$ and we choose arbitrary $x_i \in R_i$. The conditions $f = g + \sum f_i$, $f_i = 0$ outside Q_i, and $\int f_i \, d\mu = 0$ hold by definition. It is easy to check that $Q_i \subset B(x_i, 32 r_i)$. It is also easy to check that $|Q_i^*| \le 2|Q_i|$ so

$$\sum |Q_i^*| \le 2|\tilde{R}_i| \le \frac{8\|f\|_{L^1}}{\lambda} .$$

Clearly

$$\sum_i \int |f_i| \le 2 \sum_i \int |h_i| \le 2 \int |f| = 2\|f\|_{L^1} .$$

Now, g is equal to f outside $E = \bigcup \tilde{R}_i$, and

$$g = \sum_i \left(\sum_x h_i(x) q^{l(x)} \right) \frac{\chi_{R_i}}{|R_i|}$$

on E. Since $(Mf)(x) \le \lambda$ outside E, a fortiori $g(x) = f(x) \le \lambda$ outside E. On E we observe that the R_i are disjoint so

$$\sup_{x \in E} |g|(x) \le \sup_i \left| \frac{\sum_x h_i(x) q^{l(x)}}{|R_i|} \right| \le \sup_i \frac{6q\lambda |\tilde{R}_i|}{|R_i|} \le 24 q \lambda$$

which ends the proof. □

4. Integral kernels on the weighted tree

Let $(Lf)(x) = \sum_{d(x,y)=1} f(y)$. The spectral resolution of L is given by the formula:

$$(F(L)f)(x) = \frac{q}{2\pi(q+1)} \int_0^\pi F(2\sqrt{q}\cos\theta) \sum_y \frac{\phi_\theta(x,y)}{|c_\theta|^2} f(y)\, d\theta$$

where

$$\phi_\theta(x,y) = q^{-d(x,y)/2} \left(c_s e^{i\theta d(x,y)} + \bar{c}_s e^{-i\theta d(x,y)} \right) = q^{-d(x,y)/2} 2\Re(c_s e^{i\theta d(x,y)})$$

and

$$c_s = \frac{-qi}{q+1} \frac{1}{2\sin\theta} \left(e^{i\theta} - \frac{1}{q} e^{-i\theta} \right) .$$

When q is odd (so the tree may by identified with a free group) the formula is given in [8] as Theorem 4.1 (one must change notation: $q \mapsto 2r-1$, $\theta \mapsto \log(2r-1)t$). For even q the formula (and the proof) in [8] still works. Let $q^{l/2}$ stand for the operator $f(x) \mapsto q^{l(x)/2} f(x)$.

We have $2\sqrt{q}q^{l/2}A = Lq^{l/2}$ so

$$
\begin{aligned}
(F(A)f)(x) = \frac{q}{2\pi(q+1)} \int_0^\pi & F(\cos\theta) q^{-l(x)/2} \\
& \times \sum_y q^{l(y)/2} q^{-d(x,y)/2} 2\Re(c_s e^{i\theta d(x,y)}) \frac{1}{|c_\theta|^2} f(y)\, d\theta \ .
\end{aligned}
$$

To simplify the sequel we put $\tilde{A} = I - A$. For our purpose we may treat the real and imaginary parts of F separately, so we may assume that F is real. Then

$$
\begin{aligned}
F(\tilde{A})(x,y) &= \Re\left(K(x,y) \int_0^\pi F(1-\cos\theta) e^{i\theta d(x,y)} \eta(\theta) \sin\theta \, d\theta\right) \\
&= \Re(K(x,y) E_F(d(x,y)))
\end{aligned}
$$

where

$$
K(x,y) = q^{(-l(y)-l(x)-d(x,y))/2} \ ,
$$

$$
\begin{aligned}
\eta(\theta) = \frac{q}{2\pi(q+1)} \frac{2}{\bar{c}_s \sin\theta} &= \frac{q}{2\pi(q+1)} \frac{4(q+1)}{qi\left(e^{-i\theta} - \frac{1}{q}e^{i\theta}\right)} \\
&= \frac{2}{\pi i\left(e^{-i\theta} - \frac{1}{q}e^{i\theta}\right)}
\end{aligned}
$$

and

$$
E_F(k) = \int_0^\pi F(1-\cos\theta) e^{i\theta k} \eta(\theta) \sin\theta \, d\theta \ . \tag{1}
$$

Lemma 4.1. *Let $s \geq 0$ and let m be the integer satisfying $s \leq m$. Let $a < b$, $c < d$ be fixed constants. Suppose $\phi : \mathbb{R} \to \mathbb{R}$ is $C(m)$ and increasing with $\phi(c) < a$, $\phi(d) > b$, $\phi' > 0$. Then there is C depending only on a, b, c, d, s, $\|1/(\phi')\|_{L^\infty}$ and $\|\phi\|_{C(m)}$ such that*

$$
\|F \circ \phi\|_{H(s)} \leq C \|F\|_{H(s)} \ .
$$

whenever $\operatorname{supp} F \subset [a,b]$.

Proof. This lemma only formulates a well-known fact. □

Lemma 4.2. *Fix ε, $0 < \varepsilon \le 1$. If $s > \frac{3}{2} + \varepsilon$, then there exists $C > 0$ such that if for any nonnegative integer n we have*

$$\operatorname{supp} F \subset [2^{-2n-1}, 2^{-2n+2}] \cap [0, 3/2],$$

then

$$\sum_{k=0}^{\infty} |E_F(k)|(1+k)(1+2^{-n}k)^{\varepsilon} \le C\|\mathcal{D}_{2^{-2n}} F\|_{H(s)} \ ,$$

$$\sum_{k=0}^{\infty} |E_F(k)|(1+2^{-n}k)^{\varepsilon} \le C2^{-n}\|\mathcal{D}_{2^{-2n}} F\|_{H(s)} \ ,$$

$$\sum_{k=0}^{\infty} |E_F(k+1) - E_F(k)| \, (1+k)(1+2^{-n}k)^{\varepsilon} \le C2^{-n}\|\mathcal{D}_{2^{-2n}} F\|_{H(s)} \ .$$

Proof. Put

$$F(x) = (\mathcal{D}_{2^{2n}} G)(x) = G(2^{2n}x) \ .$$

Changing variables in equation 1 to $t = 2^n\theta$ we get

$$\begin{aligned} E_F(k) = 2^{-n} \int_1^{\pi} G(2^{2n+1}\sin^2(2^{-n-1}t))\eta(2^{-n}t)\sin(2^{-n}t)e^{i2^{-n}tk}\, dt \\ = 2^{-2n}\int_1^{\pi} H(t)\psi_l(t)e^{itm}\, dt = 2^{-2n}\int_1^{\pi} \tilde{H}_l(t)e^{itm}\, dt \end{aligned}$$

where $k = 2^n m + l$, $0 \le l < 2^n$

$$\tilde{H}_l(t) = H(t)\psi_l(t) \ ,$$

$$H(t) = G(2^{2n+1}\sin^2(2^{-n-1}t)) \ ,$$

$$\psi_l(t) = e^{i2^{-n}tl}2^n\sin(2^{-n}t)\eta(2^{-n}t) \ .$$

One easily sees that $\psi_l(t)$ and its derivatives are bounded uniformly in n, l and $t \in [0, \pi]$. Also, each of the derivatives of $\phi_n = 2^{2n+1}\sin^2(2^{-n-1}t)$ is uniformly bounded in n and ϕ_n' is uniformly bouded from below on counterimage of $\operatorname{supp} G_n$, so by 4.1

$$\|\tilde{H}_l\|_{H(s)} \le C\|H\|_{H(s)}\|\psi_l\|_{C(s)} \le C'\|G\|_{H(s)} \ .$$

Hence

$$\sum_{k=0}^{\infty} |E_F(k)|\,(1+k)(1+2^{-n}k)^{\varepsilon}$$

$$\leq 2^{n+1} \sum_{l=0}^{2^n-1} \sum_{m=0}^{\infty} |E_F(2^n m + l)|\,(1+m)^{1+\varepsilon}$$

$$\leq 2^{n+1} \sum_{l=0}^{2^n-1} \left(\sum_{m=0}^{\infty} (|E_F(2^n m + l)|\,(1+m)^s)^2 \right)^{1/2} \left(\sum_{m=0}^{\infty} (1+m)^{2(1+\varepsilon-s)} \right)^{1/2}$$

$$\leq C 2^{n+1} 2^{-2n} \sum_{l=0}^{2^n-1} \|\tilde{H}_l\|_{H(s)} \leq C'' \|G\|_{H(s)} = C'' \|\mathcal{D}_{2^{-2n}} F\|_{H(s)}\ .$$

The second estimate follows similarly. For the third estimate one should note that $e^{i(k+1)\theta} - e^{ik\theta} = (1-e^{i\theta})e^{ik\theta}$ and then the first factor contributes 2^{-n} to the final result. □

Recall that for a complex function f on the tree we define the gradient ∇f by the formula

$$(\nabla f)(x) = \sum_{w: d(x,w)=1} |f(w) - f(x)|\ .$$

The gradient so defined is a sublinear operator.

Lemma 4.3. *Suppose $d(y,z) = 1$ and $l(z) = l(y) - 1$. Then*

$$\sum_{x: d(x,y)=k} K(x,y) q^{l(x)} = \begin{cases} 1 & \text{for } k = 0 \\ 2 + \frac{q-1}{q}(k-1) & \text{for } k > 0 \end{cases}\ ,$$

$$\sum_{x: d(x,y)=k} |K(x,y) - K(x,z)| q^{l(x)} \leq 2\ ,$$

$$\sum_{x: d(x,y)=k} |\nabla_x K(x,y)| q^{l(x)} = 1 - \frac{1}{q} \leq 2\ .$$

Proof. Every vertex at distance k from y is reached starting from y making p steps up and then $k-p$ steps down. There is only one possibility for each step up, while we have q choices when making a step down. However, we should not go back on our steps, so if $p, k-p > 0$, then we have only $q-1$ possibilities for the first step down. Also, note that $l(x)-l(y)=2p-k$. Hence

$$\sum_{x:d(x,y)=k} q^{(-l(y)-l(x)-d(x,y))/2}q^{l(x)} = \sum_{x:d(x,y)=k} q^{(-l(y)+l(x)-k)/2}$$
$$= q^k q^{(-k-k)/2} + \left(\sum_{p=1}^{k}(q-1)q^{k-p-1}q^{(2p-k-k)/2}\right) + q^{(2k-k-k)/2}$$
$$= 1 + \frac{q-1}{q}\left(\sum_{p=1}^{k-1} 1\right) + 1 \ .$$

In the second sum, values of $K(x,y)$ and $K(x,z)$ are equal, unless x lies below z, so we get only the $p=0$ term in the sum over p.

Similarly, in the third sum only the $p=k$ term gives a nonzero contribution. Moreover, for the unique x contributing to the $p=k$ term, the contribution $|K(w,y)-K(x,y)|$ to $\nabla_x K(x,y)$ is zero for all but one w. For that one w, which is the unique vertex at distance one to x which lies above x, we have $l(w)=l(x)+1$, $d(w,y)=d(x,y)+1$, and $K(w,y)=\frac{1}{q}K(x,y)$. □

Lemma 4.4. *Fix ε, $0<\varepsilon\le 1$. If $s>\frac{3}{2}+\varepsilon$, then there is C such that if n is a nonnegative integer and* $\operatorname{supp} F \subset [2^{-2n-1}, 2^{-2n+2}] \cap [0,3/2]$, *then*

$$\sum_x |F(\tilde{A})(x,y)|\,(1+2^{-n}d(x,y))^{\varepsilon} q^{l(x)} \le C\|\mathcal{D}_{2^{-2n}}F\|_{H(s)} \ ,$$
$$\sum_x |F(\tilde{A})(x,y) - F(\tilde{A})(x,z)| q^{l(x)} \le C2^{-n} d(y,z)\|\mathcal{D}_{2^{-n}}F\|_{H(s)} \ ,$$
$$\sum_x |\nabla_x F(\tilde{A})|(x,y)\,(1+2^{-n}d(x,y))^{\varepsilon} q^{l(x)} \le C2^{-n}\|\mathcal{D}_{2^{-n}}F\|_{H(s)} \ .$$

Proof. We will prove only the second inequality (the first is easier, the third is similar to the second). It is enough to prove the lemma for real F. We may assume that $d(y,z)=1$ and that $l(z)=l(y)-1$.

Then

$$\begin{aligned}
&\sum_x |F(\tilde{A})(x,y) - F(\tilde{A})(x,z)|q^{l(x)} \\
&\qquad = \sum_x |\Re(K(x,y)E_F(d(x,y)) - K(x,z)E_F(d(x,z)))|q^{l(x)} \\
&\qquad\qquad \le \sum_x |K(x,y) - K(x,z)||E_F(d(x,y))|q^{l(x)} \\
&\qquad\qquad + \sum_x K(x,z)|E_F(d(x,y)) - E_F(d(x,z))|q^{l(x)} = S_1 + S_2 \ .
\end{aligned}$$

Now, by 4.2 and 4.3

$$\begin{aligned}
S_1 &= \sum_k |E_F(k)| \sum_{x:d(x,y)=k} |K(x,y) - K(x,z)|q^{l(x)} \\
&\qquad \le 2\sum_k |E_F(k)| \le C2^{-n}\|\mathcal{D}_{2^{-n}}F\|_{H(s)} \ .
\end{aligned}$$

To estimate the second sum note that if $d(x,z) = k$ then $d(x,y) = k \pm 1$, so $|E_F(d(x,y)) - E_F(d(x,z))| \le |E_F(k+1) - E_F(k)| + |E_F(k) - E_F(k-1)|$ (formally putting $E_F(-1) = E_F(0)$). Then (again using 4.2 and 4.3)

$$\begin{aligned}
S_2 &\le \sum_k (|E_F(k+1) - E_F(k)| + |E_F(k) - E_F(k-1)|) \sum_{x:d(x,z)=k} K(x,z)q^{l(x)} \\
&\le 4\sum_k |E_F(k+1) - E_F(k)|\,(1+k) \\
&\le C2^{-n}\|\mathcal{D}_{2^{-n}}F\|_{H(s)} \ . \quad \square
\end{aligned}$$

Proof of 2.3. Let $(-1)^l$ represent the operator $f(x) \mapsto (-1)^{l(x)}f(x)$. Observe that $-(-1)^l Af = A((-1)^l f)$. That is, A is conjugated to $-A$ by an operator which preserves all $L^p(d\mu)$-norms as well as the weak-$L^1(d\mu)$ measure of the size of a function. Consequently, it is sufficient to prove the first claim of the theorem in the case $F(\lambda) = H(1-\lambda)$.

By standard estimates, the principal hypothesis on H is independent of the choice of ϕ. Now fix nonnegative $\phi, \psi \in C_c^\infty([1/2, 4])$ such that

$\phi = \psi^2$, and for all $x > 0$ $\sum_{n=-\infty}^{\infty} \phi(2^{2n}x) = 1$. We have

$$H(x) = \sum_{n=-\infty}^{\infty} G_n(x) = \sum_{n=0}^{\infty} G_n(x) \qquad \text{where} \qquad G_n(x) = \phi(2^{2n}x)H(x)\ .$$

Of course

$$H(\tilde{A}) = \sum_{n=0}^{\infty} G_n(\tilde{A})\ .$$

We are going to show that the $G_n(\tilde{A})$ satisfy the assumptions of 1.2. By hypothesis, there is $s > \frac{3}{2}$ and C independent of n such that

$$\|\mathcal{D}_{2^{-2n}} G_n\|_{H(s)} \leq C\ .$$

Fix ε, $0 < \varepsilon \leq 1$, such that $s > \frac{3}{2} + \varepsilon$. By 4.4

$$\begin{aligned}&\int |G_n(\tilde{A})|(x,y)\,(1 + 2^{-n}d(x,y))^{\varepsilon}\,d\mu(x)\\ &\quad = \sum_x |G_n(\tilde{A})(x,y)|\,(1 + 2^{-n}d(x,y))^{\varepsilon} q^{l(x)} \leq C\|\mathcal{D}_{2^{-2n}} G_n\|_{H(s)} \leq C'\end{aligned}$$

so the first assumption of 1.2 is satisfied. Similarly the second assumption of 1.2 follows directly from 4.4 so we get first claim of 2.3.

To get the second claim write

$$\frac{1}{\sqrt{t}} = \sum_n \frac{\phi(2^{2n}t)}{\sqrt{t}} = \sum_n U_n(t) = \sum_n \frac{\psi(2^{2n}t)}{\sqrt{t}} \psi(2^{2n}t) = \sum_n V_n(t) W_n(t)\ .$$

It is easy to see that (for any $s > 0$)

$$\|\mathcal{D}_{2^{-2n}} U_n\|_{H(s)} = \|2^n U_0\|_{H(s)} = C2^n\ ,$$

$$\|\mathcal{D}_{2^{-2n}} V_n\|_{H(s)} = \|2^n V_0\|_{H(s)} = C2^n\ ,$$

and

$$\|\mathcal{D}_{2^{-2n}} W_n\|_{H(s)} = \|W_0\|_{H(s)} = C\ .$$

Using the third part of 4.4 we have for some $C = C(s)$

$$\int |\nabla_x V_n(\tilde{A})(x,y)|\,d\mu(x) \leq C$$

so using the second part of 4.4 we get

$$\int |\nabla U_n(\tilde{A})(x,y) - \nabla U_n(\tilde{A})(x,z)|\, d\mu(x)$$
$$\leq \left(\sup_w \int |\nabla V_n(\tilde{A})(x,w)|\, d\mu(x)\right)$$
$$\times \left(\int |W_n(\tilde{A})(w,y) - W_n(\tilde{A})(w,z)|\, d\mu(w)\right) \leq C' 2^{-n} d(y,z)\ .$$

In this way we have checked that

$$\nabla U_n(\tilde{A})$$

satisfies the second assumption of 1.2. We verify the first assumption of 1.2 by direct application of 4.4 to $U_n(t)$. This ends the proof. □

5. A covering lemma on certain AN groups

Let $G = \mathbb{R} \ltimes N$, $N = \mathbb{R}^n$. We assume that the multiplication is given by the formula

$$(t, x_1, \ldots, x_n)(s, y_1, \ldots, y_n)$$
$$= (t+s, \exp(a_1 s)x_1 + y_1, \ldots, \exp(a_n s)x_n + y_n)$$

where the $a_i \neq 0$ are real numbers. In the sequel we will pretend that a_i are positive, negative a_i are easy but tedious to handle. One easily checks that Lebesgue measure on $\mathbb{R}^{n+1}$ is left invariant, so we may take it as Haar measure. On G we consider the natural right-invariant riemanianian distance given by $ds^2 = dt^2 + \sum \exp(-2ta_i)\, dx_i^2$. Put $M = 2\max(1, a_1, \ldots, a_n)$. Note that for large balls (say $r > 1$) we have

$$\{(t,x) : |x| < ce^c r, |t| < cr\} \subset B(0,r) \subset \{(t,x) : |x| < e^{Mr}, |t| < r\}$$

Lemma 5.1. *G has the Calderón–Zygmund property.*

We need some preparation before the proof. We say that a parallelopiped R is admissible iff $(t,x) \in R$ if and only if

$$t \in [m_0 2^{k_0}, (m_0+1)2^{k_0}) \ ,$$

$$x_i \in [m_i 2^{k_i}, (m_i+1)2^{k_i}) \qquad \text{for } i = 1,\dots,n$$

where m_i, k_i for $i = 0,\dots,n$ are integers and for $k_0 < 0$ we have

$$e^{2M} 2^{k_0} \le \exp(-a_i(m_0 + \frac{1}{2})2^{k_0})2^{k_i} \le 4e^{8M}2^{k_0}$$

$i = 1,\dots,n$ and for $k_0 \ge 0$ we have

$$\exp(M2^{k_0+1}) \le \exp(-a_i(m_0+\frac{1}{2})2^{k_0})2^{k_i} \le 4\exp(M2^{k_0+3}) \ .$$

We will write $R = R(k_0,\dots,k_n,m_0,\dots,m_n)$.

Note that there is C such that if $R = R(k_0,\dots,k_n,m_0,\dots,m_n)$ is an admissible parallelopiped, x_R is the centerpoint of R and $r_R = 2^{k_0}$, then $R \subset B(x_R, Cr_R)$ and $|R^*| = |\{x : d(x,R) < r_R\}| \le 32^Q$.

Lemma 5.2. *If Q is an admissible parallelopiped, then there exists a sequence of partitions $\mathcal{P}_j$ of Q such that*

- *each $\mathcal{P}_j$ consists of admissible parallelopipeds,*
- *for each j all $R \in \mathcal{P}_j$ have a common k_0*
- *for each $R \in \mathcal{P}_j$ either $R \in \mathcal{P}_{j+1}$ or R is a sum of two members of $\mathcal{P}_{j+1}$ (of equal volumes).*
- *the parallelopipeds in $\mathcal{P}_j$ are arbitrarily small for large j.*

Proof. We write

$$\mathcal{P}_j = \{R(k_0,\dots,k_n,m_0,\dots,m_n) \subset Q : k_0 = f_0(j) \ ,$$
$$k_i = f_i(j,m_0), i = 1,\dots,n\}$$

where f_i are to be specified. Now, the second condition is satisfied by definition. To satisfy the third condition we require that either

$f_0(j+1) = f_0(j)$ and $f_i(j, m_0) - 1 \leq f_i(j+1, m_0) \leq f_i(j, m_0)$ or $f_0(j+1) = f_0(j) - 1$ and $f_i(j+1, 2m_0) = f_i(j+1, 2m_0+1) = f_i(j, m_0)$. The first case correspond to splitting (some of) parallelopipeds in x coordinates, the second corresponds to splitting all parallelopipeds into half in t coordinate. As a normalization we also require that $f_0(0) = k_0$, that when splitting in x we perform as many splittings as allowed by admissibility condition, and that $\mathcal{P}_j \neq \mathcal{P}_{j+1}$.

To finish the proof we should show how to divide $\mathcal{P}_j$. If $f_0(j) \leq 0$, then in each step we just substract one from f_i (cycling over i, $i = 0, \dots, n$). So in the sequel we need only to handle $f_0(j) > 0$. Our rules make as keep $f_0(j)$ unchanged and decrease $f_i(j, m_0)$, $i = 1, \dots, n$ as long as admissibility allows. So, we may assume that splitting in x coordinates is forbidden. Hence, we have

$$\exp(M2^{f_0(j)+1}) \leq \exp(-a_i(m_0 + \frac{1}{2})2^{f_0(j)})2^{f_i(j,m_0)} \leq 2\exp(M2^{f_0(j)+1})$$

Now according to our rules, we make $f_0(j+1) = f_0(j) - 1$, $f_i(j+1, 2m_0) = f_i(j+1, 2m_0+1) = f_i(j, m_0)$ and we should check that the result is admissible. However

$$\begin{aligned}\exp(-a_i(2m_0 + \frac{1}{2})2^{f_0(j+1)}) &= \exp(-a_i(m_0 + \frac{1}{4})2^{f_0(j)}) \\ &\geq \exp(-a_i(m_0 + \frac{1}{2})2^{f_0(j)})\exp(-M2^{f_0(j)})\end{aligned}$$

so

$$\begin{aligned}&\exp(-a_i(2m_0 + \frac{1}{2})2^{f_0(j+1)})2^{f_i(j+1,2m_0)} \\ &\qquad \geq \exp(-M2^{f_0(j)})\exp(-a_i(m_0 + \frac{1}{2})2^{f_0(j)})2^{f_i(j,m_0)} \\ &\qquad \geq \exp(-M2^{f_0(j)})\exp(M2^{f_0(j)+1}) \\ &\qquad = \exp(M2^{f_0(j)}) = \exp(M2^{f_0(j+1)+1})\end{aligned}$$

which gives one of admissibility conditions (lower bound for $2m_0$). Similar, computation gives upper bound for $2m_0+1$, and two other bounds (upper bound for $2m_0$ and lower bound for $2m_0+1$). □

Proof of 5.1. Fix f and $\lambda > 0$. We should decompose f. First, note that there is a partition $\mathcal{P}$ of G into dyadic parallelopipeds such that for each $Q \in \mathcal{P}$ we have $|Q| > \|f\|_{L^1}/\lambda$. Namely, let $\psi_i(l, m_0)$ be the largest integers so that $R(l, \psi_1(l, m_0), \ldots, \psi_n(l, m_0), m_0, m_1, \ldots, m_n)$ is admissible. We take l large enough and enough and write

$$G = \bigcup_{m_i\in\mathbb{Z}, m_0\in\{0,-1\}} R(l, \psi_1(l, m_0), \ldots, \psi_n(l, m_0), \ldots, m_0, m_1, \ldots, m_n)$$
$$\cup \bigcup_{k\geq l, m_i\in\mathbb{Z}, m_0\in\{1,-2\}} R(k, \psi_1(k, m_0), \ldots, \psi_n(k, m_0), m_0, m_1, \ldots, m_n)\,.$$

Next, on each $Q \in \mathcal{P}$ we apply 5.2 and use standard stopping time argument. □

6. Integral kernels on Lie groups

On a Lie group with right-invariant distance function $d(\cdot, \cdot)$, let $d(x) = d(x, e)$, so $d(x, y) = d(xy^{-1})$.

Theorem 6.1. *Assume G and L are as in* [22]. *If $\varepsilon > 0$, if $s_0, s_1 > \frac{3}{2} + \varepsilon$, if $s_1 > \frac{Q+1}{2} + \varepsilon$, then there is C such that for* $\operatorname{supp} F \subset [1/2, 2]$ *and $t \geq 1$*

$$\int |F(tL)|(x)\,(1 + t^{-1/2}d(x))^{\varepsilon} \leq C\|F\|_{H(s_0)}$$

while for $0 < t \leq 1$

$$\int |F(tL)|(x)\,(1 + t^{-1/2}d(x))^{\varepsilon} \leq C\|F\|_{H(s_1)}\,.$$

Proof. This follows using the methods of [22]. □

In particular, 6.1 applies to the group in the statement of Theorem 2.4, namely $G = \mathbb{R} \ltimes \mathbb{R}^Q$ where $s \in \mathbb{R}$ acts on $\mathbb{R}^Q$ via $n \mapsto e^{-s}n$. For our next result, 6.2, our attention will be exclusively on that group.

We recall several conventions relative to convolution and the modular function. As always, convolution of measures is defined by

$$\int_G f(x)\, d(\mu_1 * \mu_2)(x) = \int_{G\times G} f(yz)\, d\mu_1(y)\, d\mu_2(z)\ .$$

The $*$-operation on measures is $d\mu^*(x) = d\bar{\mu}(x^{-1})$. One has $(\mu_1 * \mu_2)^* = \mu_2^* * \mu_1^*$. For Dirac measures $\delta_x * \delta_y = \delta_{xy}$ and $(\delta_x)^* = \delta_{x^{-1}}$. If $\|\mu\|$ stands for the total variation, then $\|\mu_1 * \mu_2\| \leq \|\mu_1\|\,\|\mu_2\|$, $\|\mu^*\| = \|\mu\|$.

The base measure on G, denoted simply by dx, is taken as *left-invariant* Haar measure, and thus d^*x is right-invariant Haar measure. The modular function is defined by $d^*x = \delta(x)\, dx$. It follows that

$$\begin{aligned} d(xx_0) &= d^*(x_0^{-1}x^{-1}) = \delta(x_0^{-1}x^{-1})\, d(x_0^{-1}x^{-1}) \\ &= \delta(x_0^{-1}x^{-1})\, d(x^{-1}) = \delta(x_0^{-1}x^{-1})\, d^*(x) = \delta(x_0^{-1})\, d(x)\ . \end{aligned}$$

For the group under consideration, $\mathbb{R} \ltimes \mathbb{R}^Q$, one finds that dx is ordinary Lebesgue measure and $\delta(s, n) = e^{-Qs}$.

For the purposes of convolution and the $*$-operation, we identify the function $f(x)$ with the measure $f(x)\, dx$. One then finds:

$$\begin{aligned} (f_1 * f_2)(x) &= \int_G f_1(y) f_2(y^{-1}x)\, dy\ , \quad & (\delta_x * f)(z) &= f(x^{-1}z)\ , \\ f^*(x) &= \delta(x)\bar{f}(x^{-1})\ , & (f * \delta_x)(z) &= \delta(x) f(zx^{-1})\ . \end{aligned}$$

Having fixed this identification it is automatic that $(f_1 * f_2)^* = f_2^* * f_1^*$, $(f * \delta_x)^* = \delta_{x^{-1}} * f^*$, etc. Since L^1-norm (relative to dx) corresponds to total variation, one has $\|f_1 * f_2\|_{L^1} \leq \|f_1\|_{L^1}\|f_2\|_{L^1}$, $\|f^*\|_{L^1} = \|f\|_{L^1}$. The inner product for $L^2(dx)$ may be written as $\langle f_1, f_2\rangle = (f_2^* * f_1)(e)$.

As explained in Section 2 we work with the *right-invariant* vector fields X_i, and the right-invariant Laplacian L given by

$$X_0 = \partial_s \;, \qquad X_i = e^s \partial_{n_i},\; i = 1, \dots, Q \;, \qquad L = -\sum_{0 \le i \le Q} X_i^2 \;.$$

On $L^2(dx)$ this Laplacian is symmetric and may be extended to a positive self-adjoint operator. Let p_t be the *heat kernel* for (G, L), meaning that $\exp(-tL)(f) = p_t * f$. Since $\exp(-tL)$ is self-adjoint, one has $p_t = p_t^*$.

We use the (right-invariant) distance function adapted to the vector fields X_i, $0 \le i \le Q$. Here follows the usual definition. For fixed $x, y \in G$ consider all smooth curves $\gamma : [0,1] \to G$ such that $\gamma(0) = x$, $\gamma(1) = y$. For such a curve write $\gamma'(t) = \sum_i a_i(t) X_i$. Then

$$d(x,y) = \inf_\gamma \left(\int_0^1 \sum_i |a_i(s)|^2 \, ds \right)^{1/2} \;.$$

It follows that $|(X_i d)(x)| \le 1$ in the Lipschitz sense. For the group under consideration $d(x)$ is smooth (except at $x = e$), so this inequality is valid in the naive sense (except at $x = e$).

Theorem 6.2. *Assume $G = \mathbb{R} \ltimes \mathbb{R}^Q$ and L are as in 2.4. There exist C and $\varepsilon > 0$ such that uniformly in $t > 0$*

$$\sum_i \int |X_i p_t(x)| \exp(\varepsilon t^{-1/2} d(x)) \, dx \le C t^{-1/2} \;.$$

Proof. As long as t is bounded from above the estimate follows easily from well-known pointwise bounds on the heat kernel, see for example [18] (in fact [18] is an overkill, since for elliptic operators the estimates we need where already known in the sixties). So it is enough to prove our claim for $t > 1$.

We may assume that $Q = 2l$ is even. Indeed G_Q is a quotient of G_{Q+1}, the heat kernel on G_Q is the push-forward of the heat kernel on G_{Q+1}, and for $0 \le i \le Q$ the vector fields (X_i) on G_Q and G_{Q+1} respectively match up under the quotient map. Consequently our estimate on G_{Q+1} implies the same estimate on G_Q.

With $Q = 2l$ one gets

$$\delta(s,n) = e^{-2ls}\ , \qquad\qquad \delta(s,n)^{1/2} = e^{-ls}\ .$$

The distance on G is given by the formula

$$d((s,n),e) = \operatorname{arc\,cosh}\big(\tfrac{1}{2}(e^s + e^{-s}(1+|n|^2))\big)\ . \tag{2}$$

The following explicit formula for $p_t(s,n)$ is taken from [5].

$$\begin{aligned} p_t(x) &= e^{ct}\delta^{1/2} q_t(d(x,e))\ , \\ q_t(r) &= Ce^{-ct}t^{-1/2} D_r^l \exp(-r^2/(4t)) \end{aligned}$$

where

$$D_r = \frac{-1}{\sinh(r)}\partial_r\ .$$

The reduction to even Q avoids the use of fractional derivatives in the above formula.

The first stage of our proof follows along the lines of [5]. Put

$$\Phi_1(r) = \frac{r}{\sinh(r)}\ , \qquad\qquad \Phi_{j+1} = D_r\Phi_j\ .$$

Easy induction shows that one can write

$$CD_r^l \exp(-r^2/(4t)) = \exp(-r^2/(4t)) \sum_{k=1}^{l} t^{-k}\psi_{l,k}(r)\ ,$$

$$\psi_{l,k} = \sum_{|\alpha|=l} c_\alpha \prod_{i=1}^{k} \Phi_{\alpha_i}$$

where the c_α are positive.

One then checks that when $r \to \infty$

$$\Phi_j(r) = re^{-jr} + O(e^{-jr}) ,$$
$$\partial_r \Phi_j(r) = \partial_r(re^{-jr}) + O(e^{-jr})$$

so

$$\psi_{l,k}(r) = r^k e^{-lr} + O((r+1)^{k-1}e^{-lr}) ,$$
$$\partial_r \psi_{l,k}(r) = \partial_r(r^k e^{-lr}) + O((r+1)^{k-1}e^{-lr}) .$$

It is known that the Φ_j are all positive. So, likewise, all the $\psi_{l,k}$ are positive. Now the formula for the heat kernel reads

$$p_t = \sum_{k=1}^{l} t^{-(2k+1)/2} \exp(-d^2/(4t)) \delta^{1/2} \psi_{l,k}(d) . \tag{3}$$

Hence

$$\begin{aligned} X_i p_t = {} & t^{-3/2} \exp(-d^2/(4t)) X_i(\delta^{1/2}\psi_{l,1}(d)) \\ & + (X_i d/2) t^{-5/2} d \exp(-d^2/(4t)) \delta^{1/2} \psi_{l,1}(d) \\ & + X_i \left(\sum_{k=2}^{l} t^{-(2k+1)/2} \exp(-d^2/(4t)) \delta^{1/2} \psi_{l,k}(d) \right) . \end{aligned} \tag{4}$$

We will deal with these three terms separately. The first term (which we will deal with last) is the really delicate one. To bound the second term, we use the inequality $|X_i d| \leq 1$:

$$\begin{aligned} |(X_i d/2) t^{-5/2} d \exp(-d^2/(4t)) \delta^{1/2} \psi_{l,1}(d)| & \\ \leq C t^{-2} (d^2/(4t))^{1/2} \exp(-d^2/(4t)) \delta^{1/2} \psi_{l,1}(d) & \\ \leq C' t^{-2} \exp(-d^2/(8t)) \delta^{1/2} \psi_{l,1}(d) & \\ \leq C'' t^{-1/2} p_{2t} . & \end{aligned}$$

Now use the fact that $\|p_{2t}\|_{L^1} = 1$.

Although the third term of (4) is given by a complicated formula, it is really of lower order (in t) than the others, and can be bounded quite directly. Consider the term for some $k \geq 2$.

$$
\begin{aligned}
&|X_i \big(t^{-(2k+1)/2} \exp(-d^2/(4t))\delta^{1/2}\psi_{l,k}(d)\big)\,| \\
&\qquad = |t^{-(2k+1)/2}\big((X_i \exp(-d^2/(4t)))\delta^{1/2}\psi_{l,k}(d) \\
&\qquad\qquad + \exp(-d^2/(4t))(X_i\delta^{1/2})\psi_{l,k}(d) \\
&\qquad\qquad + \exp(-d^2/(4t))\delta^{1/2}(X_i\psi_{l,k}(d)\big)| \\
&\qquad = |t^{-(2k+3)/2}(X_i d/2) d \exp(-d^2/(4t))\delta^{1/2}\psi_{l,k}(d) \\
&\qquad + t^{-(2k+1)/2}\big(\exp(-d^2/(4t))(X_i\delta(e))\delta^{1/2}\psi_{l,k}(d) \\
&\qquad\qquad + \exp(-d^2/(4t))\delta^{1/2}(X_i d)\partial_r\psi_{l,k}(d)\big)| \\
&\qquad \leq Ct^{-(2k+3)/2} d \exp(-d^2/(4t))\delta^{1/2}(d+1)^k e^{-ld} \\
&\qquad + Ct^{-(2k+1)/2} \exp(-d^2/(4t))\delta^{1/2}(d+1)^k e^{-ld} \\
&\leq C' t^{-2}(1 + d^k t^{-k/2} + d^{k-1}t^{-(k-1)/2}) \exp(-d^2/(4t))\delta^{1/2}(d+1)e^{-ld} \\
&\qquad \leq C'' t^{-2} \exp(-d^2/(8t))\delta^{1/2}(d+1)e^{-ld} \leq C''' t^{-1/2} p_{2t} \,.
\end{aligned}
$$

At the next to the last line, we use $k \geq 2$ to get t^{-2}.

For the first term of (4), where $k = 1$, we must give a more detailed argument. In that term occurs the factor $X_i(\delta^{1/2}\psi_{l,1}(d))$ where $\delta^{1/2} = \exp(-ls)$ and $\psi_{l,1}(d) \approx cd\exp(-ld)$. Except for a relatively small piece of G, the two exponentials, $\exp(-ls)$ and $\exp(-ld)$, almost cancel one another out, and so the derivative $X_i(\delta^{1/2}\psi_{l,1}(d))$ is very much smaller than one would naively expect.

A comparison with the analogous calculation on a tree is striking.

$$
\begin{aligned}
&\delta^{1/2}\exp(-d) &&\longleftrightarrow && K(x,y)\,, \\
&\delta^{1/2} &&\longleftrightarrow && q^{(-l(x)-l(y))/2}\,, \\
&\exp(-d) &&\longleftrightarrow && q^{-d(x,y)/2}\,.
\end{aligned}
$$

In the case of the tree the two exponential factors cancel each other out perfectly (and consequently $\nabla_x K(x,y) = 0$) unless x lies directly above y. In the present case, things are fuzzier, but essentially the same phenomenon controls the situation.

For $T > 1$, $\operatorname{arc\,cosh}(T) = \log(2T) + (1/4)T^{-2} + \dots$, the expansion continuing as a convergent power series in T^{-2}. Applying this to (2), that is to the exact formula for $d(s,n) = d((s,n),e)$ gives

$$d(s,n) = \log(e^s + e^{-s}(1+|n|^2)) + O((e^s + e^{-s}(1+|n|^2))^{-2}) \, ,$$

$$\partial_s d(s,n) = \partial_s \log(e^s + e^{-s}(1+|n|^2)) + O((e^s + e^{-s}(1+|n|^2))^{-2}) \, ,$$

$$e^s \partial_{n_i} d(s,n) = e^s \partial_{n_i} \log(e^s + e^{-s}(1+|n|^2)) + O((e^s + e^{-s}(1+|n|^2))^{-2}) \, .$$

Next, since $\delta^{1/2}(s,n) = e^{-ls}$, since $\psi_{l,1}(r) = cre^{-lr} + O(e^{-lr})$, and since $\partial_r \psi_{l,1}(r) = -clre^{-lr} + O(e^{-lr})$:

$$\begin{aligned}
(\delta^{1/2}\psi_{l,1})(s,n) &= cd(s,n)(e^{2s} + 1 + |n|^2)^{-l} + O(\delta^{1/2}e^{-ld}) \, , \\
\partial_s(\delta^{1/2}\psi_{l,1})(s,n) &= cd\partial_s(e^{2s} + 1 + |n|^2)^{-l} + O(\delta^{1/2}e^{-ld}) \\
&= cd\frac{-2le^{2s}}{(e^{2s} + 1 + |n|^2)^{l+1}} + O(\delta^{1/2}e^{-ld}) \, ,
\end{aligned}$$

and

$$\begin{aligned}
e^s\partial_{n_i}(\delta^{1/2}\psi_{l,1})(s,n) &= cd\partial_{n_i}(e^{2s} + 1 + |n|^2)^{-l} + O(\delta^{1/2}e^{-ld}) \\
&= cd\frac{-2le^{s}n_i}{(e^{2s} + 1 + |n|^2)^{l+1}} + O(\delta^{1/2}e^{-ld}) \, .
\end{aligned}$$

Now

$$\int |\partial_s p_t|(x) \exp(\varepsilon t^{-1/2} d(x))\, dx$$
$$\le C \int |t^{-3/2} \exp(-d^2/(4t)) d \partial_s((e^{2s} + 1 + |n|^2)^{-l})| \exp(\varepsilon t^{-1/2} d(x))\, dx$$
$$+ C t^{-1/2} \int p_{2t} \exp(\varepsilon t^{-1/2} d(x))\, dx$$
$$+ t^{-3/2} \int \exp(-d^2/(4t)) O(\delta^{1/2} e^{-ld}) \exp(\varepsilon t^{-1/2} d(x))\, dx$$
$$= C(I_1 + I_2) + I_3 \ .$$

Similarly

$$\int |e^s \partial_{n_i} p_t|(x) \exp(\varepsilon t^{-1/2} d(x))\, dx \le C(I_4 + I_2) + I_3$$

where

$$I_4 = \int |t^{-3/2} \exp(-d^2/(4t)) d e^s \partial_{n_i}((e^{2s}+1+|n|^2)^{-l})| \exp(\varepsilon t^{-1/2} d(x))\, dx \ .$$

So to finish the proof we need to estimate I_i, $i = 1, 2, 3, 4$. We can absorb $\exp(\varepsilon t^{-1/2} d(x))$ into the Gaussian factor, so

$$I_2 \le C t^{-1/2} \int p_{3t} = C t^{-1/2} \ .$$

We can compare I_3 with a linear combination of the p_t, using the bound

$$O(\delta^{1/2} e^{-ld}) \le C 2^{2k} p_{2^{2k}} \qquad \text{for } 2^k \le d(x) < 2^{k+1}.$$

This follows from (3), the formula for p_t, recalling that all the terms there are positive. Hence

$$
\begin{aligned}
t^{-3/2} &\int \exp(-d^2/(4t))O(\delta^{1/2}e^{-ld})\exp(\varepsilon t^{-1/2}d(x))\,dx \\
&= t^{-3/2}\int_{d(x)\geq t^{1/2}} + t^{-3/2}\int_{d(x)<t^{1/2}} \\
&\leq Ct^{-1/2}\int p_{2t} + t^{-3/2}\int_{d(x)<t^{1/2}} O(\delta^{1/2}e^{-ld}) \\
&\leq Ct^{-1/2} + t^{-3/2}\int_{d(x)<1} O(\delta^{1/2}e^{-ld}) \\
&\quad + t^{-3/2}\sum_{k\geq 0, 2^k<t^{1/2}} \int_{2^k\leq d(x)<2^{k+1}} O(\delta^{1/2}e^{-ld}) \\
&\leq C't^{-1/2} + \sum_{k\geq 0, 2^k<t^{1/2}} C''t^{-3/2}2^{2k} \leq C't^{-1/2} + C'''t^{-1/2}\,.
\end{aligned}
$$

To estimate I_1 note that

$$
\int_{\mathbb{R}^{2l}} \frac{2le^{2s}}{(e^{2s}+1+|n|^2)^{l+1}}\,dn = \int_{\mathbb{R}^{2l}} \frac{2l}{(1+e^{-2s}+|n|^2)^{l+1}}\,dn \leq C
$$

with constant C independent of s. Hence

$$
\begin{aligned}
I_1 &= \int |t^{-3/2}\exp(-d^2/(4t))d\partial_s((e^{2s}+1+|n|^2)^{-l})|\exp(\varepsilon t^{-1/2}d(x))\,dx \\
&\leq Ct^{-1}\int |\exp(-d^2/(8t))\partial_s((e^{2s}+1+|n|^2)^{-l})|\,dx \\
&\leq Ct^{-1}\int \exp(-s^2/(8t))\int_{\mathbb{R}^{2l}} \frac{2le^{2s}}{(e^{2s}+1+|n|^2)^{l+1}}\,dn\,ds \\
&\leq C't^{-1}\int \exp(-s^2/(8t))\,ds = C''t^{-1/2}\,.
\end{aligned}
$$

The argument for I_4 is similar, and that ends the proof. □

The key estimate 6.2 has now been proved for the case of $G = \mathbb{R}\ltimes\mathbb{R}^Q$, as needed for the proof of 2.4. We will state our next two results in greater generality, using that key estimate as one of the hypotheses.

We recall the general set-up. Let G be a Lie group. On G use left-invariant Haar measure as the basic measure. Work always with right-invariant vector fields and differential operators. Assume that on G there are given vector fields X_i and that the distance $d(\cdot,\cdot)$ on G is the distance adapted to those vector fields. Assume also that there is a given self-adjoint Laplacian $L \geq 0$ on G, and that its heat kernel is $p_t(\cdot)$.

Lemma 6.3. *Let G, X_i, d, L, and p_t be as above. If, as in 6.2, we have $\|X_i p_t\|_{L^1} \leq C t^{-1/2}$ for each i, then there is C' such that*

$$\|p_t * (\delta_x - \delta_y)\|_{L^1} \leq C' t^{-1/2} d(x,y) \ .$$

Proof. First, we need an auxiliary formula. Let $\gamma : [0,1] \mapsto G$ be a smooth curve. Fix s. Assume that $\gamma'(s) = Y(\gamma(s))$, where Y is a right-invariant vector field. We have

$$\partial_u (p_t * \delta_{\gamma(s+u)})|_{u=0} = \partial_u (p_t * \delta_{\exp(uY)} * \delta_{\gamma(s)})|_{u=0}$$

so, applying the $*$-operation and using the fact that $p_t^* = p_t$,

$$\|\partial_u (p_t * \delta_{\gamma(s+u)})|_{u=0}\|_{L^1} = \|\partial_u (\delta_{(\gamma(s))^{-1}} * \delta_{\exp(-uY)} * p_t)|_{u=0}\|_{L^1} = \|Y p_t\|_{L^1} \ .$$

If $\gamma'(s) = \sum_i a_i(s) X_i$, then since s was arbitrary,

$$\|\partial_s (p_t * \delta_{\gamma(s)})\|_{L^1} \leq \sum_i |a_i(s)| \, \|X_i p_t\|_{L^1} \ .$$

Now, assume that γ joins x and y. Then

$$\begin{aligned} &\|p_t * (\delta_x - \delta_y)\|_{L^1} \\ &\qquad \leq \int_0^1 \|\partial_s (p_t * \delta_{\gamma(s)})\|_{L^1} \, ds \leq \int_0^1 \sum_i |a_i(s)| \|X_i p_t\|_{L^1} \, ds \\ &\qquad\qquad \leq C t^{-1/2} \left(\int_0^1 \sum_i |a_i(s)|^2 \, ds \right)^{1/2} \ . \end{aligned}$$

Since $d(x,y) = \inf_\gamma (\int_0^1 \sum_i |a_i(s)|^2 \, ds)^{1/2}$ we get the claim. □

Theorem 6.4. *Let G, X_i, d, L, and p_t be as above. Suppose that*

- *G satisfies the Calderón–Zygmund condition,*
- *the conlusion of 6.2 holds.*

Then the Riesz transforms $X_i L^{-1/2}$ are of weak type $(1,1)$ and bounded on L^p, $1 < p \leq 2$.

Proof. We are going to use 1.2. We write

$$\Gamma(\tfrac{1}{2}) X_i L^{-1/2} = \int_0^\infty t^{-1/2} X_i p_t \, dt = \sum_n \int_{2^n}^{2^{n+1}} t^{-1/2} X_i p_t \, dt = \sum_n K_n \ .$$

For our next calculation we use the following natural convention. Suppose K is the operator given by the kernel $K(x,y)$, namely $(Kf)(x) = \int K(x,y) f(y)\, dy$. Then for any measure μ we define

$$(K\mu)(x) = \int K(x,y)\, d\mu(y).$$

Note, that $((f \cdot g) * \delta_y)(x) = (f * \delta_y)(x) g(xy^{-1})$. Now, putting $a = 1$ and $c = 2^{-1/2}$

$$\begin{aligned}
&\int |K_n(x,y)(1 + c^{-n} d(x,y))| \, dx \\
&\quad \leq \int_{2^n}^{2^{n+1}} \int |(t^{-1/2}(X_i p_t) * \delta_y)(x)(1 + c^{-n} d(xy^{-1}))| \, dx \\
&\qquad = \int_{2^n}^{2^{n+1}} \|(t^{-1/2}(X_i p_t)(1 + c^{-n} d)) * \delta_y\|_{L^1} \, dt \\
&\qquad \leq C \int_{2^n}^{2^{n+1}} \|t^{-1/2}(X_i p_t) \exp(\varepsilon t^{-1/2} d)\|_{L^1} \, dt \\
&\qquad\qquad \leq C' \int_{2^n}^{2^{n+1}} t^{-1} \, dt \leq C'
\end{aligned}$$

so the first assumption about K_n in 1.2 holds.

$$
\begin{aligned}
\int |K_n(x,y) - K_n(x,z)|\,dx &= \|K_n(\delta_y - \delta_z)\|_{L^1} \\
&\leq \int_{2^n}^{2^{n+1}} \|t^{-1/2}(X_i p_t)(\delta_y - \delta_z)\|_{L^1}\,dt \\
&\leq \int_{2^n}^{2^{n+1}} t^{-1/2}\|X_i p_{t/2}\|_{L^1}\,\|p_{t/2} * (\delta_y - \delta_z)\|_{L^1}\,dt \\
&\leq C\int_{2^n}^{2^{n+1}} t^{-3/2} d(y,z)\,dt \leq C' 2^{-n/2} d(y,z) = C' c^n d(y,z)
\end{aligned}
$$

so the second assumption of 1.2 is also satisfied, ending the proof. □

Proof of 2.4. The result will follow from 1.2. Indeed, by 5.1, G has the Calderón–Zygmund property. Fix $\phi \in C_c^\infty(\mathbb{R}_+)$ such that $\operatorname{supp}\phi \subset [1/2, 2]$ and $\sum_n \phi(2^n x) = 1$ for all $x > 0$. Put $F_n(x) = F(2^{-n}x)\phi$ and $G_n(x) = F_n(x)\exp(x)$. We write

$$F(L) = \sum F_n(2^n L) = \sum G_n(2^n L)\exp(-2^n L)$$

Now the first assumption of 1.2 follow from 6.1 (applied directly to F_n) and the second from 6.1 (applied to G_n) and 6.3, which ends the proof. □

This proof of the spectral multiplier part of 2.4 is also applicable to any G, X_i, d, L, and p_t as above such that

- G satisfies the Calderón–Zygmund condition,
- the conclusion of 6.1 holds,
- the conclusion of 6.3 holds.

7. Final remarks

The method used to obtain the Calderón–Zygmund decomposition on the tree is inspired by [32] (p. 309, Lemma XVII.3.2) and uses the same ideas as [17]. Working on trees made ideas simpler, and made clear which maximal function is relevant for singular integrals (on

Lie groups there are many natural maximal functions, some bounded, some unbounded [9],[10]). We use a somewhat different method on Lie groups, which allows us to handle the case where the roots are of both signs (for instance, unimodular groups). Both proofs are related to the construction of Fölner sequences.

For simplicity, we restricted ourselves to rank 1 case. However, our arguments works the same in a product setting.

The first author can prove that if a (locally compact) group G with left-invariant Haar measure and right-invariant distance satisfies the Calderón–Zygmund property, then G is amenable.

In [27] Sjögren proves that $X_1 L^{-1/2}$ is of weak type $(1,1)$ in the $Q = 1$ case, $G = \mathbb{R} \ltimes \mathbb{R}$. The result in our Theorem 2.4 is stronger than that because it deals also with X_0. In [31] Wängeforos extends Sjögren's results to AN groups associated to arbitrary rank 1 symmetric spaces. In [14] Gaudry and Sjögren prove, again for $G = \mathbb{R} \ltimes \mathbb{R}$, that $L^{-1/2} X_1$ is of weak type $(1,1)$. That operator is $-(X_1 L^{-1/2})^*$, so their result gives $p > 2$ estimates for $X_1 L^{-1/2}$. Although Gaudry and Sjögren do not mention trees explicitly, Theorem 3 in [14] does have a strong tree-like flavor.

References

[1] F. Astengo, Multipliers for a distinguished Laplacean on solvable extensions of H-type groups, Monatsh. Math. 120 (1995), 179–188.

[2] M. Christ, D. Müller, On L^p spectral multipliers for a solvable Lie group, Geom. Funct. Anal. 6 (1996), 860–876.

[3] J. L. Clerc, E. M. Stein, L^p-multipliers for noncompact symmetric spaces. *Proc. Nat. Acad. Sci. U.S.A.* 71 (1974), 3911–3912.

[4] M. Cowling, Harmonic analysis on semigroups, Ann. of Math. 117 (1983), 267–283.

[5] M. Cowling, G. Gaudry, S. Giulini, G. Mauceri, Weak type $(1,1)$ estimates for heat kernel maximal functions on Lie groups. *Trans. Amer. Math. Soc.* 323, (1991), 637–649.

[6] M. Cowling, S. Giulini, A. Hulanicki, G. Mauceri, Spectral multipliers for a distinguished Laplacian on certain groups of exponential growth, *Studia Math.* 111 (1994), 103–121.

[7] E. Damek, A. Hulanicki, J. Zienkiewicz, Estimates for the Poisson kernels and their derivatives on rank one NA groups. *Studia Math.* 126 (1997), no. 2, 115–148.

[8] A. Figa-Talamanca, M. Picardello, Harmonic analysis on free groups, Dekker, New York 1983.

[9] G. Gaudry, S. Giulini, A. Hulanicki, A. M. Mantero, Hardy-Littlewood maximal functions on some solvable Lie groups. *J. Austral. Math. Soc. Ser. A* 45 (1988), no. 1, 78–82.

[10] G. Gaudry, S. Giulini, and A. M. Mantero, Asymmetry of maximal functions on the affine group of the line. *Tôhoku Math. J.* 42 (1990), 195–203.

[11] G. Gaudry, T. Qian, P. Sjögren, Singular integrals associated to the Laplacian on the affine group $ax + b$. *Ark. Mat.* 30 (1992), no. 2, 259–281.

[12] G. Gaudry, P. Sjögren, Singular integrals on Iwasawa NA groups of rank 1. *J. Reine Angew. Math.* 479 (1996), 39–66.

[13] G. Gaudry, P. Sjögren, Singular integrals on the complex affine group. *Colloq. Math.* 75 (1998), no. 1, 133–148.

[14] G. Gaudry, P. Sjögren, Haar-like expansions and boundedness of a Riesz operator on a solvable Lie group. *Math. Z.* 232 (1999), no. 2, 241–256.

[15] S. Giulini, G. Mauceri, Analysis of a distinguished Laplacian on solvable Lie groups. *Math. Nachr.* 163 (1993), 151–162.

[16] S. Giulini, G. Mauceri, S. Meda, L^p multipliers on noncompact symmetric spaces. *J. Reine Angew. Math.* 482, (1997), 151–175.

[17] S. Giulini, P. Sjögren, A note on maximal functions on a solvable Lie group. *Arch. Math. (Basel)* 55 (1990), no. 2, 156–160.

[18] W. Hebisch, Estimate on the semigroups generated by left invariant operators on Lie groups, J. reine angew. Math. 423 (1992), 1–45.

[19] W. Hebisch, The subalgebra of $L^1(AN)$ generated by the laplacean, Proc. AMS 117 (1993), 547–549.

[20] W. Hebisch, Boundedness of L^1 spectral multipliers for an exponential solvable Lie group, *Coll. Math.* 73 (1997), 155–164.

[21] W. Hebisch, Spectral multipliers on exponential growth solvable Lie groups, *Math. Z.* 229 (1998), 435–441.

[22] W. Hebisch,Analysis of Laplacians on solvable Lie groups, Notes for ICMS conference Edinburg 1998.

[23] N, Lohoué, S. Mustapha, Sur les transformées de Riesz sur les groupes de Lie moyennables et sur certains espaces homogènes, Canad. J. Math. 50 (1998), 1090–1104.

[24] J. Ludwig, D. Müller, Sub-Laplacians of holomorphic L^p-type on rank one AN-groups and related solvable groups, J. of Funct. Anal. 170 (2000), 366-427.

[25] S. Mustapha, Multiplicateurs spectraux sur certains groupes non-unimodulaires, Harmonic Analysis and Number Theory, CMS Conf. Proceedings, Vol 21, 1997.

[26] S. Mustapha, Multiplicateurs de Mikhlin pour une classe particulière de groupes non-unimodulaires, Ann. Inst. Fourier 48 (1998), 957–966.

[27] P. Sjögren, An estimate for a first-order Riesz operator on the affine group. *Trans. Amer. Math. Soc.* 351 (1999), no. 8, 3301–3314.

[28] J.-O. Strömberg, Weak type L^1 estimates for maximal functions on noncompact symmetric spaces. *Ann. of Math.* (2) 114 (1981), no. 1, 115–126.

[29] E. M. Stein, Topics in harmonic analysis related to the Littlewood-Paley theory, Ann. of Math. Stud. 63, Princeton Univ. Press, Princeton 1970.

[30] M. Taylor, L^p -Estimate on functions of the Laplace operator, *Duke Math. J.* 58 (1989), 773-793.

[31] M. Wängeforos, An estimate for some first-order Riesz operators on Iwasawa NA groups of rank 1, preprint 1997

[32] A. Zygmund, Trigonomeric series, Cambridge University Press 1968.

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