On operators satisfying Rockland condition

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Abstract

Let G be a homogeneous Lie group. We prove that for every closed, homogeneous subset Γ of G^* which is invariant under the coadjoint action, there exists a regular kernel P such that P goes to 0 in any representation from Γ and P satisfy Rockland condition outside Γ. We prove a subelliptic estimate as an application.

Introduction

The purpose of this paper is to construct operators which satisfy Rockland condition in a given set of representations Γ, and are equal to 0 outside Γ. Rockland operators satisfy remarkable subelliptic estimates ([11], [7], [9], [10], [14] see also [15]) making them good substitute for elliptic operators on homogeneous groups. Christ et al. [2] gave a calculus for pseudodifferential operators on homogeneous groups: the formulas for products and adjoints and criteria for existence of left or right parametrices (generalizing results of [8]). However, one should note that great flexibility of classical calculus of pseudodifferential operators is in large part due to the ease of constructing scalar functions (cutoffs and partitions of unity). In homogeneous group case we want to pre-specify operators in a set of representations and still have a regular kernels — this is not straightforward — in fact not always possible. Our kernels may serve as cutoffs on spectral side (for the spatial cutoffs one simply uses multiplications with smooth functions). The conditions we impose seem to be necessary. We present also a simple application in which we derive some L^p estimates.

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Preliminaries

We consider a homogeneous group G, that is a nilpotent Lie group equipped with a family of automorphisms (*dilations*) $\{\delta_t\}_{t>0}$ such that $\delta_t \delta_s = \delta_{ts}$ and for all $x \in G$ we have

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$\delta_t x \to e$ if $t \to 0$. The reader may wish to consult [6] (our definition is a bit more general). We will identify G with its Lie algebra via the exponential map, and write 0 instead of e. With our identification all δ_t became linear maps.

As $\det(\delta_t)$ must be a power of t there exists a number $Q > 0$ such that for all bounded measurable $A \subset G$

$$|\delta_t A| = t^Q |A|,$$

this Q is called the homogeneous dimension of G. More general, one can take t to be discrete, that is consider dilation operator D such that $D^{-k}x \to e$ if $k \to \infty$.

A distribution T on G is said to be a kernel of order $r \in \mathbb{C}$ if T coincides with a locally finite measure away from the origin, and is homogeneous of degree $-r - Q$, that is satisfies

$$(f \circ \delta_t, T) = t^r (f, T)$$

for all $f \in C_c^\infty(G)$ and $t > 0$. We extend action of dilation to distributions by the formula

$$(f, \delta_t T) = (f \circ \delta_t, T).$$

Then T is a kernel of order r iff for all $t > 0$

$$\delta_t T = t^r T.$$

A kernel is called regular if it coincides with a smooth function away from the origin.

In the sequel we will identify right-invariant vector fields on G with distributions supported in $\{0\}$. More precisely, there is one to one correspondence between right-invariant differential operators and distributions supported in $\{0\}$. To get the identification we write

$$(X, f) = Xf(0).$$

Then $Xf = X * f$, and for the left-invariant field $\tilde{X}$ corresponding to X we have $\tilde{X} f = f * X$. We also note that dilating a vector field as an element of Lie algebra and as a distribution gives the same result.

For a unitary representation π of G on a Hilbert space H and a kernel T of order r, $\Re(r) > 0$, the operator $\pi(T)$ is defined on the space $C^\infty(\pi)$ of smooth vectors for π by

$$(g, \pi(T) f) = (\phi_{f,g}, T)$$

where $\phi_{f,g}(x) = (g, \pi(x)f)$. Equivalent definition is:

$$\pi(T)f = T * \psi_f(e)$$

where $\psi_f(x) = \pi(x)f$. This definition also makes sense for uniformly bounded representations on Banach spaces. If $\Re(r) \leq 0$ the situation is more tricky. For a regular kernel T one may find $h \in C_c^\infty(G)$ such that

$$Tf = \sum_k 2^{-rk}\delta_{2^k}(h) * f.$$

For $r = 0$ one must have $\int h = 0$ (otherwise T would not be a distribution), and the Cotlar-Stein lemma shows that the sum defining T is strongly convergent in any unitary representation of G to a bounded operator.

If $\Re(r) < 0$ then T defines unbounded operator on $L^2(G)$ (to see that T is densely defined see [2]). The following lemma shows that the problem is caused by trivial representation.

(1.1). **Lemma**. *Let G be a homogeneous nilpotent Lie group, π be nontrivial irreducible unitary representation of G and h be a Schwartz class function on G. Then for any m there is a continuous seminorm $C_m(\cdot)$ such that $\|\pi(\delta_t(h))\| \leq C_m(h)(1+t)^{-m}$.*

Proof. We fix a scalar product (so also a norm $|\cdot|$) on the Lie algebra of G. Note, that there exists $\alpha > 0$ such that if X is an element of Lie algebra of G then $|\delta_{t^{-1}}(X)| \leq C(1+t)^{-\alpha}$ for $t > 1$. Let X_j span the Lie algebra of G. Put $L = \sum X_j^2$. It is known that L is invertible in π. So

$$\mathrm{id}_\pi = \sum \pi(X_j)(\pi(X_j)(\pi(L)^{-1})) = \sum \pi(X_j)E_j$$

where $E_j = \pi(X_j)\pi(L)^{-1}$ are bounded operators in π. Next, we write

$$\pi(\delta_t(h)) = \pi(\delta_t(h))\sum \pi(X_j)E_j = \sum \pi(\delta_t(h * \delta_{t^{-1}}(X_j)))E_j.$$

Inductively, for any natural l

$$\pi(\delta_t(h)) = \sum \pi(\delta_t(h * \delta_{t^{-1}}(X_{j_1}) * \ldots * \delta_{t^{-1}}(X_{j_l})))E_{j_l} \cdot \ldots \cdot E_{j_1}$$

so

$$\|\pi(\delta_t(h))\| \leq C_l \max \|\pi(\delta_t(h * \delta_{t^{-1}}(X_{j_1}) * \ldots * \delta_{t^{-1}}(X_{j_l})))\| \leq$$

$$C_l \max \|h * \delta_{t^{-1}}(X_{j_1}) * \ldots * \delta_{t^{-1}}(X_{j_l})\|_{L^1} \leq$$

$$C_l \max C_{h,l} |\delta_{t^{-1}}(X_{j_1})| \cdot \ldots \cdot |\delta_{t^{-1}}(X_{j_l})| \leq$$

$$C'(h,l)(1+t)^{-\alpha l}$$

which gives the claim.

If we put

$$V_s = \{f \in L^1_{loc}(G) \cap C^\infty(G - \{0\}) : \forall_{\phi \in C_c^\infty(G-\{0\})} \lim_{t\to 0} t^s \phi \delta_t f = 0;$$

$$\text{uniformly with all derivatives}\}$$

then V_s is a locally convex metrizable vector space and Schwartz class functions are dense in V_s. For $0 > \Re(r) > -s$ regular kernels of order r are in V_s and (1.1) shows that π has (unique) extension form Schwartz class to V_s.

Main Results

Let π_l for $l \in G^*$ be the representation associated with l according to Kirilov theory (cf. [13]).

(2.2). **Theorem**. *Let G be a homogeneous Lie group with dilations $\{\delta_t\}_{t>0}$, and Γ be a closed subset of G^* such that $Ad^*(G)\Gamma \subset \Gamma$, $\forall_{t>0}\delta_t\Gamma \subset \Gamma$. For every $\alpha \geq 0$ there exists a regular kernel P of order α such that for all $l \in \Gamma$ we have $\pi_l(P) = 0$ and for all $l \notin \Gamma$ the operator $\overline{\pi_l(P)}$ is positive definite and injective on its domain. For every $0 > \alpha > -Q$ there exists a kernel satisfying conditions above, except for $l = 0$. Moreover, there is a Schwartz class function H on G such that for all $l \in \Gamma$ we have $\pi_l(H) = 0$ and for all $l \notin \Gamma$ the operator $\pi_l(H)$ is positive definite and injective.*

Proof. It is enough to prove the theorem only for $-Q < \alpha \leq 0$ and small $\alpha > 0$. Indeed, taking sufficiently high power of P we get α as large as we wish, without destroying other properties of P. Moreover, we only need to prove the last claim, that is to construct a

Schwartz class function H such that for all $l \in \Gamma$ we have $\pi_l(H) = 0$ and for all $l \notin \Gamma$ the operator $\pi_l(H)$ is positive definite and injective. If we have such a function and α is small enough, then

$$P = \int_0^\infty t^{-\alpha} \delta_t H \frac{dt}{t}$$

give as a regular kernel of order α having the required properties. Indeed, 0 is in Γ so $\int H = 0$. Moreover there is $\alpha_0 > 0$ such that for all $0 < t < 1$ we have $|\delta_t x| \leq t^{\alpha_0}|x|$. Fix a $\phi \in C_c^\infty(G)$. For $0 < t < 1$ we have

$$|(\phi, \delta_t H)| = |(\phi - \phi(0), \delta_t H)| \leq C t^{\alpha_0}$$

so if $\alpha < \alpha_0$, then

$$\int_0^\infty t^{-\alpha}(\phi, \delta_t H)\frac{dt}{t}$$

is absolutely convergent. Changing variables in the integral above one easily checks that P is homogeneous of degree $-\alpha - Q$. Smoothness of P outside 0 is clear. Also, if $\alpha > 0$, it is easy to check that

$$\pi(P) \subset \int_0^\infty t^{-\alpha}\pi(\delta_t H)\frac{dt}{t}$$

and that the right hand side defines closed injective operator (we compute the integral applying function under the integral to a vector — the domain is the set of all vectors for which the integral is convergent). For $\alpha \leq 0$ we get the same conclusion using (1.1) (condition $\alpha > -Q$ is used to prove that P is a regular kernel).

We are going to build H. Let us recall that by [1] the set $\tilde{\Gamma} = \{\pi_l : l \in \Gamma\}$ is closed in the Fell topology of the space of representations. Fix $p \notin \Gamma$. $Ad^*(G)\Gamma \subset \Gamma$ implies $\pi_p \notin \tilde{\Gamma}$. By the definition of Fell topology and density of $C_c^\infty(G)$ in $C^*(G)$, there exists a function $F \in C_c^\infty(G)$ such that

$$\|\pi_p(F)\| = 1$$

and for all $l \in \Gamma$

$$\|\pi_l(F)\| < 1/10.$$

Replacing F by $F^* * F$ we may assume that F is positive definite. Choose $\phi \in C_c^\infty(\mathbb{R})$ such that $\phi(1) = 1$, $\phi \geq 0$, $\text{supp}\ (\phi) \subset [1/10, 2]$. By the spectral theorem the operator

$$\phi(\pi_p(F)) \neq 0$$

while for all $l \in \Gamma$

$$\phi(\pi_l(F)) = 0.$$

Using functional calculus, as for example in [12], we show that there exists a Schwartz class function R on G such that (as a convolution operator on $L^2(G)$) $R = \phi(F)$. Approximating ϕ by polynomials we see that for all l we have

$$\phi(\pi_l(F)) = \pi_l(R).$$

We also note that the set of π such that $\pi(R) \neq 0$ is open (by definition). To summarize, we constructed R such that for all $\pi \in \tilde{\Gamma}$

$$\pi(R) = 0,$$

the set $U_R = \{\pi(R) \neq 0\}$ is open and $\pi_p \in U_R$. Since Fell topology has a countable basis, there exists a sequence $\{R_i\}_{i \in \mathbb{N}}$ such that the complement of $\tilde{\Gamma}$ is the union of U_{R_i}. Therefore, putting $S = \sum a_i R_i$ where a_i are positive and small enough for S to be a Schwartz class function we see that for all π in $\tilde{\Gamma}$

$$\pi(S) = 0$$

and $\pi(S) \neq 0$ on the complement of $\tilde{\Gamma}$.

To finish the proof we need the following lemma:

(2.3). **Lemma**. *If π is an irreducible unitary representation of G, the sequence $\{g_j\}_{j \in \mathbb{N}}$ is dense in G, $f \in L^1(G)$, $\pi(f) \neq 0$, $\pi(f) \geq 0$, the sequence $\{c_j\}_{j \in \mathbb{N}}$ is positive and summable, then*

$$A = \pi\left(\sum c_j \delta_{g_j^{-1}} f \delta_{g_j}\right)$$

(where δ_{g_j} means convolution operator with unit mass at g_j) is injective.

Proof. Suppose, on the contrary that A is not injective. Then, there exists a nonzero v such that

$$(Av, v) = \sum c_j(\pi(\delta_{g_j^{-1}})\pi(f)\pi(\delta_{g_j})v, v) = 0,$$

hence for each j

$$(\pi(f)^{1/2}\pi(\delta_{g_j})v, \pi(f)^{1/2}\pi(\delta_{g_j})v) = 0,$$

or simply

$$\pi(f)^{1/2}\pi(\delta_{g_j})v = 0.$$

Since π is irreducible, closed linear span of $\pi(\delta_{g_j})v$ gives the whole space, so $\pi(f)^{1/2} = 0$. As $\pi(f)$ is nonzero this gives a contradiction.

Choosing $c_j = \exp(-j - |g_j|)$ and applying (2.3) we conclude that

$$H = \sum c_j \delta_{g_j^{-1}} S \delta_{g_j}$$

is a Schwartz class function such that for π in the complement of $\tilde{\Gamma}$ the operator $\pi(H)$ is injective and for π in $\tilde{\Gamma}$

$$\pi(H) = 0$$

which ends the proof.

Remark If P is a regular kernel of order α, $\Gamma = \{l : \pi_l(P) = 0\}$, then $Ad^*(G)\Gamma \subset \Gamma$, $\forall_{t>0}\delta_t\Gamma \subset \Gamma$ and Γ is closed. More precisely, if $\Re\alpha < 0$, then $\Gamma - \{0\}$ is closed in $G^* - \{0\}$. The first condition is clear. Γ is invariant under dilations because P is homogeneous. To see that Γ is closed assume first that $\Re\alpha > 0$. Let ϕ_n be an approximate unit in L^1 consisting of C_c^∞ functions. $\phi_n * P \in L^1$ so $\Gamma_n = \{l : \pi_l(\phi_n * P) = 0\}$ is closed. As $\Gamma = \bigcap_n \Gamma_n$ we see that Γ is closed. If $\Re\alpha \leq 0$, then we compose P with a kernel R such that $\pi_l(R)$ is injective on smooth vectors for all $l \neq 0$ and $P * R$ have order with positive real part.

An application

As an application of our construction we will give an extension to L^p of a theorem by J. Nourrigat ([14] Théorème 1.3). We need some setup to state the theorem. Let

Ω be measure space with measure μ. Assume G act on Ω preserving the measure. Let $\phi : G \times \Omega \mapsto \mathbb{C}$ be a (measurable) cocycle for this action, that is $|\phi| = 1$ and for all $g_1, g_2 \in G$ and all $x \in \Omega$

$$\phi(g_1 g_2, x) = \phi(g_1, x)\phi(g_2, g_1^{-1}x).$$

Then the formula

$$\pi(g)f(x) = \phi(g, x)f(g^{-1}x)$$

gives continuous representation of G which act trough isometries on $L^p(\Omega)$, $1 \leq p < \infty$ (on L^∞ we get isometries, but the action is only weak-$*$ continuous). We say that π is a *cocycle representation*. The set of smooth vectors $C^\infty(\pi)$ is defined as usual (of course it may depend on p). Let us note that $\pi(C_c^\infty(G))(L^1 \cap L^\infty)$ is dense in $C^\infty(\pi)$ so we may do all the calculations on the common core. We also note that the usual construction of π_l gives cocycle representation so we may consider π_l as representations on L^p. Let us also sketch the proof of the following well-known lemma:

(3.4). **Lemma**. *If W is a regular kernel of order α, $\Re(\alpha) = 0$ and W gives bounded operator on $L^2(G)$ then W gives bounded operator on $L^p(G)$, $1 < p < \infty$.*

Remark In fact, regular kernel W of order α, $\Re(\alpha) = 0$, is always bounded on L^2.

Proof. This follows from [3] Chapitre III Théorème (2.4). G equipped with homogeneous norm is space of homogeneous type. One may easily verify that in [3] the assumption that the kernel K is in L^2 is only used to prove that the operator T is *associated* to the kernel, that is that $Tf(x) = \int K(x,y)f(y)dy$ for x not in support of f.

Alternative approach is to regularize W. We fix $\phi \in C_c^\infty(G)$ such that $\int \phi = 1$ and we write $\phi_t = \delta_t \phi$. One may check that for the regularized kernels $W_t = (\phi_t - \phi_{t^{-1}}) * W$ assumptions of [3] Chapitre III Théorème (2.4) holds with bounds on independent of regularization (and $\lim_{t\to\infty} W_t f = Wf$ for f in $L^p(G)$, $1 < p < \infty$). This approach shows that transference principle is applicable to W.

(3.5). **Theorem**. *Let G be a homogeneous Lie group with dilations $\{\delta_t\}_{t>0}$, and Γ be a closed subset of G^* such that $Ad^*(G)\Gamma \subset \Gamma$, $\forall_{t>0}\delta_t\Gamma \subset \Gamma$. Let π be a cocycle representation of G such that all irreducible components of π are of the form π_l with $l \in \Gamma$. Let R be* *a regular kernel of order α, $\Re(\alpha) > 0$ such that for all $l \in \Gamma$, $l \neq 0$ the operator $\pi_l(R)$ is injective on C^∞ vectors of π_l. Then for every $1 < p < \infty$ and every positive integer k and every kernel A of order β, $0 \leq \Re(\beta) \leq k\Re(\alpha)$ there exists $C_{p,k,A}$ such that*

$$\forall_{f \in C^\infty(\pi)} \|\pi(A)f\|_{L^p} \leq C_{p,k,A}(\|f\|_{L^p} + \|\pi(R)^k f\|_{L^p}).$$

If R is of order α, $0 \leq \alpha < Q$, then there exists a regular kernel B if order $-\alpha$ such that

$$\forall_{l \in \Gamma - \{0\}} \forall_{f \in C^\infty(\pi_l)} \pi_l(B)\pi_l(R)f = f.$$

Proof. First, assume that $k = 1$ and $\beta = \alpha$. Let S be a regular kernel of order $2\Re(\alpha)$ given by (2.2) . We put

$$T = S + R^*R.$$

T is a regular kernel of order $2\Re(\alpha)$ and the image of T in any nontrivial representation of G is injective on smooth vectors. We are going to construct an inverse of T. There exists injective positive definite operator P on $L^2(G)$ such that for $s > -Q$ operator P^s is given by the regular kernel of order s, for small $s > 0$ P^s generates a semigroup of symmetric probability measures (see [2] Theorem 6.1 and [7]). Choose $s_0 < Q$ and $m \in \mathbb{N}$ such that $s_0 m = 2\Re(\alpha)$. Put $V = P^{s_0}$ and $U = V^{-m}T$. One easily checks that U is given by a regular kernel of order 0 and that the image of U in any nontrivial irreducible unitary representation of G is injective on smooth vectors, so (by [2] Theorem 6.2) U is left invertible on $L^2(G)$ and the inverse is given by a regular kernel U^{-1}. Now the last claim follows if we notice that $U^{-1}V^{-m}R^*$ is a regular kernel of order $-\alpha$ for $\Re(\alpha) < Q$ and that

$$\pi(U^{-1}V^{-m}R^*)\pi(R)f = \pi(U^{-1})\pi(V^{-1})^m\pi(R^*)\pi(R)f$$

$$= \pi(U^{-1})\pi(V^{-1})^m\pi(T)f = \pi(U^{-1}V^{-m}T)f = f.$$

Also the operator $AT^{-1}R^*$ is given by the regular kernel $W = AU^{-1}V^{-m}R^*$ of order 0. The operator obtained from W is bounded on $L^p(G)$ (by (3.4)) so by the transference principle [4] the image of W in π is bounded on $L^p(\pi)$. Hence

$$\pi(A)f = \pi(A)\pi(U^{-1}V^{-m}T)f = \pi(A)\pi(U^{-1})\pi(V^{-1})^m\pi(S + R^*R)f$$

$$= \pi(A)\pi(U^{-1})\pi(V^{-1})^m\pi(R^*)\pi(R)f = \pi(AU^{-1}V^{-m}R^*)\pi(R)f = \pi(W)\pi(R)f$$

and

$$\|\pi(A)f\| \le \|\pi(W)\|\|\pi(R)f\|.$$

If $\Re(\beta) = \Re(\alpha)$, then

$$\|\pi(A)f\| \le C\|\pi(P^\beta)f\| \le C\|P^{\beta-\alpha}\|_{L^p(G),L^p(G)}\|\pi(P^\alpha)f\| \le C'\|\pi(R)f\|.$$

$\|P^{\beta-\alpha}\|_{L^p(G),L^p(G)}$ is finite by (3.4) .

If $\beta < \alpha$ then we note that $A(1+P^\alpha)^{-1}$ is a convolution with a L^1 function (this follows from estimates in [5]) so

$$\|Af\| \le C\|(1+P^\alpha f\| \le C'(\|f\| + \|Rf\|).$$

If $k > 1$ we simply replace R by R^k.

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