On the relation between elliptic and parabolic Harnack inequalities

W. Hebisch*

Institute of Mathematics, Wrocław University, Wrocław, Poland

L. Saloff-Coste

Department of mathematics, Cornell University, Ithaca, NY, USA

September 10, 2000

Abstract

We show that, if a certain relative Sobolev inequality holds, then a scale-invariant elliptic Harnack inequality suffices to imply its a priori stronger parabolic counterpart. Neither the relative Sobolev inequality nor the elliptic Harnack inequality alone suffices to imply the parabolic Harnack inequality in question; both are necessary conditions. As an application we show the equivalence between parabolic Harnack inequality for Δ on M, (i.e., for $\partial_t + \Delta$) and elliptic Harnack inequality for $-\partial_t^2 + \Delta$ on $\mathbb{R} \times M$.

1 Introduction

Consider the Laplace-Beltrami operator Δ on a complete Riemannian manifold (M, g) equipped with its Riemannian measure dv (our convention is that the Laplace operator has non-negative spectrum on $L^2(M, dv)$; in $\mathbb{R}^n$, $\Delta = -\sum_1^n \partial_i^2$). Solutions of the elliptic (Laplace) equation $\Delta u = 0$ (i.e., harmonic functions) and of the parabolic (heat diffusion) equation $(\partial_t + \Delta)u = 0$ (sometimes called caloric functions) are objects of intense study both because of their own significance and because their properties reflect certain aspects of the geometry of (M, g). The elliptic (resp. parabolic) scale-invariant Harnack inequality is one of the important properties that non-negative harmonic functions (resp. solutions of $(\partial_t+\Delta)u = 0$) can satisfy or not depending on the underlying Riemannian manifold (M, g).

By definition, one says that (M, g) satisfies a **scale-invariant elliptic Harnack inequality** if there exists a constant C such that for any geodesic ball $B \subset M$ and any non-negative harmonic function u in B,

$$\sup_{\frac{1}{2}B}\{u\} \leq C \inf_{\frac{1}{2}B}\{u\}. \tag{1.1}$$

Here $\frac{1}{2}B$ denote the ball concentric with B with radius half that of B. We call this inequality scale-invariant because the constant C does not depend of the radius of the ball B (nor of its center for that matter).

The parabolic counterpart of this inequality is slightly more complicated and goes as follows. One says that (M, g) satisfies a **scale-invariant parabolic Harnack inequality** if there exists a constant C such that for any reals r, s with $r > 0$, any $x \in M$, and any non-negative solution u of $(\partial_t + \Delta)u = 0$ in $Q = (s - r^2, s) \times B(x, r)$,

$$\sup_{Q_-}\{u\} \leq C \inf_{Q_+}\{u\} \tag{1.2}$$

*Partially supported by KBN grant 2 P03A 058 14 and European Commision via TMR network "Harmonic analysis"

where

$$\begin{aligned} Q_+ &= (s - r^2/4, s) \times B(x, r/2) \\ Q_- &= (s - 3r^2/4, s - r^2/2) \times B(x, r/2). \end{aligned}$$

Clearly, the parabolic version implies the elliptic one.

Euclidean spaces satisfy the parabolic scale-invariant Harnack inequality (1.2) whereas hyperbolic spaces do not: in hyperbolic spaces, the constant C in both (1.1) and (1.2) does explode as the radius r tends to infinity. Other examples of manifolds satisfying the parabolic (hence also the elliptic) scale-invariant Harnack inequality are manifolds having non-negative Ricci curvature [30] and Lie groups having polynomial volume growth, equipped with an invariant metric [46, 35, 48]. In fact, the parabolic Harnack inequality is fairly well understood thanks to the following theorem.

Theorem 1.1 ([21, 38]) *A Riemannian manifold (M, g) satisfies the scale-invariant parabolic Harnack inequality* (1.2) *if and only if (M, g) has the doubling volume property*

$$\forall\, x \in M, \;\; \forall, r > 0, \quad v(B(x, r)) \le D_0\, v(B(x, r/2)) \tag{1.3}$$

and satisfies the Poincaré inequality

$$\forall\, B = B(x, r) \subset M, \;\; \forall\, f \in \mathcal{C}^\infty(B(x, r)), \quad \int_B |f - f_B|^2 dv \le P_0\, r^2 \int_B |\nabla f|^2 dv \tag{1.4}$$

where f_B is the mean of f over the geodesic ball B.

Note that this theorem shows that (1.2) is preserved under bi-Lipschitz changes of metric, a fact that is not at all obvious. Whether or not this is true for (1.1) is not known.

Recently, examples of manifolds satisfying the scale-invariant **elliptic** Harnack inequality (1.1) but failing to satisfy the **parabolic** version (1.2) have been constructed. These examples are manifolds built from graphs having a fractal structure at infinity. On some examples, (1.1) holds whereas (1.4) fails; on other examples (1.1) holds whereas (1.3) fails; See [4, 15].

To state the main result of this paper, we need to introduce the following notion. One says that (M, g) satisfies a scale-invariant local Sobolev inequality if there exist $\nu > 2$ and S_0 such that, for any geodesic ball B of radius $r(B) > 0$ and volume $v(B)$, and any function $f \in \mathcal{C}_0^\infty(B)$,

$$\left(\int_B |f|^q dv \right)^{2/q} \le \frac{S_0\, r(B)^2}{v(B)^{2/\nu}} \int_B (|\nabla f|^2 + r(B)^{-2} |f|^2) dv \tag{1.5}$$

with $q = 2\nu/(\nu - 2)$. The exact values of q and ν are unimportant for our purpose. This inequality is also referred to as a **relative** Sobolev inequality (meaning, relative to the ball B).

Inequality (1.5) is a variant of the **global** Sobolev inequality

$$\forall\, f \in \mathcal{C}_0^\infty(M), \quad \|f\|_q^2 \le S_0\, \|\nabla f\|_2^2 \tag{1.6}$$

with again $q = 2\nu/(\nu - 2)$. Indeed, if (1.5) holds **and** $v(B) \approx r(B)^\nu$ then we can let r tend to infinity in (1.5). This yields (1.6). Typical manifolds satisfying (1.5) but not (1.6) are the flat manifolds $\mathbb{R}^n/\mathbb{Z}^m$ with $1 \le m \le n$ (here $\nu = n$). Some manifolds, e.g., hyperbolic spaces, satisfy (1.6) but not (1.5). Inequality (1.6) implies the volume growth lower estimate $v(B) \ge cr(B)^\nu$, [7]. Similarly (1.5) implies the relative volume bound

$$\frac{v(B)}{v(B')} \le D_1 \left(\frac{r(B)}{r(B')} \right)^\nu$$

for all B, B' with $B' \subset B$ [20]. In particular, (1.5) implies the doubling volume property (1.3). It is also known that (1.3) and (1.4) imply (1.5) for some S_0 and $\nu > 2$. See [38].

We can now state the main result of this paper.

Theorem 1.2 *Let (M, g) be a complete Riemannian manifold. Assume the scale-invariant local Sobolev inequality* (1.5) *is satisfied. Assume also the scale-invariant elliptic Harnack inequality* (1.1) *holds true. Then (M, g) satisfies the scale-invariant parabolic Harnack inequality* (1.2).

It is not hard to construct manifolds satisfying the Sobolev inequalities (1.5) and (1.6) and such that (1.1) and (1.2) fail: the connected sum of two copies of the Euclidean space $\mathbb{R}^n$, $n \geq 3$ provides such an example. See [29, 23]. Theorem 1.2 sheds some light on the rather mysterious property of satisfying a scale-invariant elliptic inequality. For instance, a corollary of Theorems 1.1 and 1.2 is the following.

Corollary 1.3 *Let (M, g) be a complete Riemannian manifold. Assume that (M, g) satisfies a scale invariant local Sobolev inequality* (1.5)*. Then the following two properties are equivalent:*

1. *The manifold (M, g) satisfies the scale-invariant elliptic Harnack inequality* (1.1).
2. *The manifold (M, g) satisfies the Poincaré inequality* (1.4)

Another interesting corollary concerns the Laplace equation in $\mathbb{R}^m \times M$ equipped with the obvious product Riemannian structure.

Corollary 1.4 *Let (M, g) be a complete Riemannian manifold and denote by $(M_{(m)}, g_{(m)})$ the complete Riemannian manifold equal to the product of an m dimensional Euclidean space with M. Then the following three properties are equivalent:*

1. *The manifold $(M_{(1)}, g_{(1)})$ satisfies the scale-invariant* **elliptic** *Harnack inequality* (1.1).
2. *The manifold (M, g) satisfies the scale-invariant* **parabolic** *Harnack inequality* (1.2).
3. *All the manifolds $(M_{(m)}, g_{(m)})$, $m \geq 1$, satisfy the scale-invariant parabolic Harnack inequality* (1.2).

All these results are developed below in the setting of local Dirichlet spaces.

In the last section of the paper, Section 5, we consider non-classical parabolic Harnack inequalities where the time-space scaling (t^2, t) is replaced by a more general one including scaling of the type (t^ω, t) for large t with $\omega > 2$. We establish an equivalence between such parabolic Harnack inequalities and certain non-classical two-sided Gaussian estimates of the heat kernel. Section 5 is very much motivated by the work of Barlow and Bass on fractals [3, 4, 5]. See also [24, 45]. It relates to the other results of this paper in the following way: if M is manifold where one of these non-classical parabolic Harnack inequalities holds true then, on the one hand M does not satisfy the classical parabolic Harnack inequality, on the other hand M does satisfy the classical elliptic Harnack inequality. Thus such manifolds provide examples showing that (1.1) and (1.2) are not equivalent properties. Constructing such examples is a rather non-trivial matter but such a construction is indicated in [4], based on a prefractal graph. Part of the difficulty in constructing such examples is that whether or not these non-classical parabolic Harnack inequalities are stable under rough-isometries (oe even quasi-isometries) is not known.

2 Background

2.1 Dirichlet spaces

One of the natural settings for the results of this paper is that of regular, strictly local Dirichlet spaces. Thus, let M be a connected locally compact separable space and let μ be a positive Radon measure on M with full support. For any open set $\Omega \subset M$, let $\mathcal{C}_0(\Omega)$ be the set of all continuous functions with compact support in Ω. Consider a regular Dirichlet form $\mathcal{E}$ with domain $\mathcal{D} \subset L^2(M, d\mu)$ and core $\mathcal{C} \subset \mathcal{D}$: a core is a subset of $\mathcal{D} \cap \mathcal{C}_0(M)$ which is dense in $\mathcal{D}$ for the norm $(\|f\|_2^2 + \mathcal{E}(f,f))^{1/2}$ and dense in $\mathcal{C}_0(M)$ for the uniform norm. A Dirichlet form is regular if it admits a core. See [18]. We also assume that $\mathcal{E}$ is strictly local: for any $u, v \in \mathcal{D}$ such that the supports of u and v are compact and v is constant in a neighborhood of the support of u, we have $\mathcal{E}(u,v) = 0$. See [18, pg 6] where such Dirichlet forms are called "strong local". Any such Dirichlet form $\mathcal{E}$ can be written in terms of an "energy measure" Γ so that $\mathcal{E}(u,v) = \int_M d\Gamma(u,v)$ where $d\Gamma(u,v)$ is a signed radon measure for $u, v \in \mathcal{D}$. Moreover, Γ satisfies the Leibnitz rule and the chain rule. See [18, pg 115-116]. In the Riemannian case, for all $u \in \mathcal{C}_0^\infty(M)$, $\Gamma(u,u)$ admits a density with respect to the Riemannian volume $d\mu = dv$ which is equal to $|\nabla u|^2$.

It is a simple but remarkable fact that the data above suffices to introduce a pseudo-distance d on M often called the **intrinsic distance** and defined as follows. Let $\mathcal{L}$ be the set of all functions f in the core $\mathcal{C}$ such that $d\Gamma(f,f) \le d\mu$, i.e., $\Gamma(f,f)$ is absolutely continuous with respect to μ with Radon-Nikodym derivative bounded by 1. In some sense, $\mathcal{L}$ is the set of all compactly supported Lipschitz functions with Lipschitz constant 1. Then, for each $x, y \in M$, define $d(x,y)$ by

$$d(x,y) = \sup\{f(x) - f(y) : f \in \mathcal{L}\}. \tag{2.1}$$

Note that d is always a lower semicontinuous function. It is only a pseudo-distance because it might happen that $d(x,y) = +\infty$ for some x, y. This actually happens in some interesting cases (see [6]) but we will not be concerned with such cases in this paper.

We now make a couple of crucial hypotheses about the Dirichlet space $(\mathcal{E}, \mathcal{D}, L^2(M, d\mu))$, in terms of the intrinsic distance d. Throughout the paper, except in Section 5, we assume the following properties are satisfied.

- The pseudo-distance d is finite everywhere and the topology induced by d is equivalent to the initial topology of M. In particular, $(x,y) \mapsto d(x,y)$ is a continuous function.
- (M, d) is a complete metric space.

These hypotheses imply that (M,d) is a path metric space (i.e., d can be defined in terms of "shortest paths"). See e.g., [25]. It also implies that the cut-off functions

$$y \mapsto \sup\{d(x,y) - r, 0\} = (d(x,y) - r)_+$$

are in $\mathcal{L}$. This is a simple but crucial fact. It allows us to extend classical arguments from the Riemannian setting to the present more general framework. For a careful introduction to the intrinsic distance and its geometry we refer the reader to [41].

We will denote by $B(x,r) = \{y \in M : d(x,y) < r\}$ the ball of radius r around x. Given a ball $B = B(x,r)$ we let $r(B) = r$ be its radius and $\mu(B)$ be its volume relative to the measure μ.

2.2 The heat semigroup

Fix a Dirichlet space $(\mathcal{E}, \mathcal{D}, L^2(M, d\mu))$ as above. As is well known, there is a self-adjoint semigroup of contractions of $L^2(M, d\mu)$, call it $(H_t)_{t>0}$, uniquely associated with this Dirichlet space. Moreover, $(H_t)_{t>0}$ is (sub-)Markovian. Let $-L$ be the infinitesimal generator of $(H_t)_{t>0}$ so that $H_t = e^{-tL}$.

We assume throughout the paper that the transition function of the semigroup $(H_t)_{t>0}$ is absolutely continuous with respect to μ. Thus, there exists a non-negative measurable function $(t,x,y) \mapsto h(t,x,y)$, the heat diffusion kernel, so that

$$H_t f(x) = \int_M h(t,x,y) f(y) d\mu(y).$$

In the present context it is useful to be a little more precise since the above formula does not uniquely define $h(t,x,y)$. In what follows, we assume that $h(t,x,y)$ is **the unique excessive**

density of H_t. (The reader unfamiliar with this notion can make the a priori restrictive assumption that $h(t,x,y)$ is continuous).

Given an open set Ω, one can easily define the heat diffusion semigroup $(H_t^\Omega)_{t>0}$ satisfying Dirichlet boundary condition in Ω. Indeed, $(H_t^\Omega)_{t>0}$ is the semigroup associated to the minimal closure of the form $\mathcal{E}$ restricted to $\mathcal{D}\cap\mathcal{C}_0(\Omega)$. We denote by $h^\Omega(t,x,y)$ the corresponding heat kernel. It is well known that, $h^\Omega(t,x,y)\le h(t,x,y)$ for all $t,x,y\in(0,+\infty)\times\Omega\times\Omega$. We also introduce the least Dirichlet eigenvalue of L in Ω by setting

$$\lambda_0(\Omega)=\inf\{\mathcal{E}(f,f): f\in\mathcal{D}\cap\mathcal{C}_0(\Omega),\ \ \|f\|_2=1\}.$$

2.3 The doubling property

Fix $R\in(0,+\infty]$. We say that $(\mathcal{E},\mathcal{D},L^2(M,d\mu))$ is R-**doubling** if for any ball B of radius less than R,

$$\mu(2B)\le D_0\,\mu(B). \tag{2.2}$$

When this holds with $R=+\infty$, we simply say that the space is **doubling**.

For later references, we note a few consequences of this volume estimate:

- If (2.2) holds,
$$\forall\, x\in M,\ \ \forall s<r<R,\ \ \frac{\mu(B(x,r))}{\mu(B(x,s))}\le D_0\left(\frac{r}{s}\right)^\nu \tag{2.3}$$
and, moreover,
$$\forall\, x,y\in M,\forall\, 0<s<r<R,\ \text{with } d(x,y)\le r,\ \ \frac{\mu(B(x,r))}{\mu(B(y,s))}\le D_1\left(\frac{r}{s}\right)^\nu \tag{2.4}$$
for any $\nu\ge\log_2 D_0$. Actually, one can take $D_1=D_0^2$.

- If a Dirichlet space is R-doubling and no ball of radius $10R$ covers M, then there exist $\beta,\gamma>0$ such that
$$\forall\, x\in M,\ \ \forall s<r<R,\ \ \frac{\mu(B(x,r))}{\mu(B(x,s))}\ge\beta\left(\frac{r}{s}\right)^\gamma. \tag{2.5}$$
See, e.g., [20, 40].

- Finally, (2.2) implies that there exists D such that
$$\forall\, T>0,\quad \mu(B(x,T))\le\mu(B(x,R))\exp\left(D(T/R)\right). \tag{2.6}$$
The above clearly follows from
$$\forall\, T>R,\quad \mu(B(x,T+R/4)\le C\mu(B(x,T)). \tag{2.7}$$
For the last inequality, note that if $\{x_i\}_{i=1}^m$ are $R/2$-net in $B(x,T-R/4)$, then the balls $B(x_i,R/4)\subset B(x,T)$ are disjoint whereas the balls $B(x_i,R)$ cover $B(x,T+R/4)$. So $\mu(B(x,T+R/4))\le\sum\mu(B(x_i,R))\le C^2\sum\mu(B(x_i,R/4))\le C^2\mu(B(x,T))$.

These elementary facts play an important role in what follows. Note in particular that R-doubling implies R'-doubling for all $R'=KR$, $K\in(1,+\infty)$. The last inequality is of special interest because, together with Grigor'yan criterion [22, Theorem 9.1] and its extension to the Dirichlet space setting [42, Theorem 4]), it implies the following result.

Theorem 2.1 *If $(\mathcal{E},\mathcal{D},L^2(M,d\mu))$ satisfies the R-doubling property* (2.2)*, then it is stochastically complete, that is, $\int_M h(t,x,y)d\mu(y)=1$ for all $t>0$ and $x\in M$.*

From a technical point of view, the main theme of this paper is heat kernel lower bounds. Hence, it is not surprising that stochastic completeness plays a role. For a survey on stochastic completeness on Riemannian manifold, see [22].

Let us also recall the following folklore lemma.

Lemma 2.2 *If $(\mathcal{E}, \mathcal{D}, L^2(M, d\mu))$ satisfies the R-doubling property* (2.2) *then, for any ball B of radius less than R, the lowest Dirichlet eigenvalue satisfies $\lambda_1(B) \leq Ar(B)^{-2}$.*

Proof This follows by a simple test function argument. □

2.4 Weak solutions and Harnack inequalities

We are interested in solutions of the elliptic and parabolic equations $Lu = 0$ and $(\partial_t + L)u = 0$. In the present context, a solution of the equation $Lu = 0$ in an open set $\Omega \subset M$ is a function u which is locally in $\mathcal{D}$ and such that for any function $\phi \in \mathcal{C} \cap \mathcal{C}_0(\Omega)$,

$$\int_\Omega d\Gamma(u, \phi) = 0. \tag{2.8}$$

We call such a solution u a **harmonic** function.

Similarly, a solution of the equation $(\partial_t + L)u = 0$ on $I \times \Omega$ (where $I \subset \mathbb{R}$ is an open interval and $\Omega \subset M$ is an open subset of M) is a measurable function $u : I \times \Omega \to \mathbb{R}$ such that $(t, x) \mapsto \partial_t u(t, x) \in L^{\infty;2}_{\text{loc}}(I \times \Omega, dt \otimes d\mu)$, $x \mapsto u(t, x) \in \mathcal{D}$ and

$$\int_M \partial_t u(t, \cdot)\phi d\mu + \int_M d\Gamma(u(t, \cdot), \phi) = 0 \tag{2.9}$$

for all $\phi \in \mathcal{C} \cap \mathcal{C}_0(\Omega)$ (it is possible to deal with solutions in a weaker sense but we will not pursue this here). For instance, for any $k = 0, 1, 2, \ldots$, the functions $(t, x) \mapsto \partial_t^k h(t, x, y)$ and $(t, y) \mapsto \partial_t^k h(t, x, y)$ are solutions of $(\partial_t + L)u = 0$ in $(0, +\infty) \times M$. Similarly, the functions $(t, x) \mapsto \partial_t^k h^\Omega(t, x, y)$ and $(t, y) \mapsto \partial_t^k h^\Omega(t, x, y)$ are solutions of $(\partial_t + L)u = 0$ in $(0, +\infty) \times \Omega$.

Fix $R \in (0, +\infty]$. We say that the Dirichlet space $(\mathcal{E}, \mathcal{D}, L^2(M, d\mu))$ satisfies a R**-scale-invariant elliptic Harnack inequality** if there exists a constant C such that for any ball B of radius $r(B) < R$ and any non-negative harmonic function u in B,

$$\sup_{\frac{1}{2}B}\{u\} \leq C \inf_{\frac{1}{2}B}\{u\}. \tag{2.10}$$

We say that the Dirichlet space $(\mathcal{E}, \mathcal{D}, L^2(M, d\mu))$ satisfies a R**-scale-invariant parabolic Harnack inequality** if there exist a constant C such that for any reals s, r with $0 < r < R$, for any ball $B = B(x, r)$, and any non-negative solution u of the equation $(\partial_t + L)u = 0$ in $Q = (s - r^2, s) \times B$,

$$\sup_{Q_-}\{u\} \leq C \inf_{Q_+}\{u\} \tag{2.11}$$

where

$$\begin{aligned} Q_+ &= (s - r^2/4, s) \times B(x, r/2) \\ Q_- &= (s - 3r^2/4, s - r^2/2) \times B(x, r/2). \end{aligned}$$

Clearly, the parabolic version implies the elliptic one.

Remark The exact value of R appearing in the Harnack inequalities above plays only a minor role: the only important distinction is between $R < +\infty$ and $R = +\infty$. This is because a R-scale-invariant elliptic (resp. parabolic) Harnack inequality implies a R'-scale-invariant elliptic (resp. parabolic) Harnack inequality for any $R' = KR$, $K \in (1, +\infty)$ with a constant C' which is a function of C and $K = R'/R$. This follows by straightforward covering arguments.

2.5 Hölder continuity

One of the important applications of the Harnack inequalities above is that they yield a certain regularity of the solution of $Lu = 0$ and $(\partial_t + L)u = 0$. This is especially noteworthy in the present framework since these solutions are not even continuous, a priori. The following are well known results: for divergence form operators in $\mathbb{R}^n$ they are due to J. Moser [31, 32] and the proofs go over to the present setting without change.

Theorem 2.3 *Fix $0 < R \le +\infty$.*

1. *Assume that the Dirichlet space $(\mathcal{E}, \mathcal{D}, L^2(M, d\mu))$ satisfies the R-scale-invariant elliptic Harnack inequality* (2.10)*. Then there exist two positive real A, α such that for any ball B of radius $r(B)$ less than R and any harmonic function u in B,*

$$\forall\, x, y \in \frac{1}{2}B, \;\; |u(x) - u(y)| \le A(d(x,y)/r)^{\alpha} \sup_B\{|u|\}. \tag{2.12}$$

2. *Assume instead that the Dirichlet space $(\mathcal{E}, \mathcal{D}, L^2(M, d\mu))$ satisfies the R-scale-invariant parabolic Harnack inequality* (2.11)*. Then there exist two positive real A, α such that for any $s \in \mathbb{R}$, any $r \in (0, R)$, any ball B of radius r, and any solution u of $(\partial_t + L)u = 0$ in $Q = (s - r^2, s) \times B$,*

$$|u(t,x) - u(\tau,y)| \le A \left[(d(x,y) + \sqrt{|t-\tau|})/r\right]^{\alpha} \sup_Q\{|u|\} \tag{2.13}$$

for all $(t,x), (\tau,y) \in (s - 4r^2/4, s - r^2/4) \times \frac{1}{2}B$.

We will refer to the properties above as R-scale-invariant Hölder regularity estimates, either elliptic or parabolic. As for Harnack inequalities, each of these estimates for a given R implies the similar estimate for all $R' = KR$, $K \in (1, +\infty)$.

2.6 Scale-invariant local Sobolev inequality

We say that the Dirichlet space $(\mathcal{E}, \mathcal{D})$ satisfies a **R-scale-invariant local Sobolev inequality** if there exists a constant S_0 and a real $\nu > 2$ such that, for any ball B of radius $r(B) \le R$ and any function $f \in \mathcal{D} \cap \mathcal{C}_0(B)$,

$$\left(\int_B |f|^q d\mu\right)^{2/q} \le \frac{S_0\, r(B)^2}{\mu(B)^{2/\nu}} \left(\int_B d\Gamma(f,f) + r(B)^{-2} \int_B |f|^2 d\mu\right) \tag{2.14}$$

where $q = 2\nu/(\nu - 2)$. The exact values of q and ν will play no role in what follows. Some authors call (2.14) a **relative Sobolev inequality**.

The following theorem gathers a number of known consequences of the Sobolev inequality (2.14).

Theorem 2.4 *Let $(\mathcal{E}, \mathcal{D}, L^2(M, d\mu))$ be a Dirichlet space such that the R-scale-invariant local Sobolev inequality* (2.14) *holds true for some fixed $R \in (0, +\infty]$. Then the following properties are satisfied:*

1. *There exists a constant $D_0 = D_0(R)$ such that, for any ball B of radius less than R,*

$$\forall\, t \in (0,1), \;\; \frac{\mu(tB)}{\mu(B)} \ge Dt^{\nu}. \tag{2.15}$$

2. *There exists $\beta, \gamma > 0$ such that for any ball B of radius less than R satisfying $10B \ne M$, we have*

$$\forall\, t \in (0,1), \;\; \frac{\mu(tB)}{\mu(B)} \le \beta t^{\gamma}. \tag{2.16}$$

3. There exists C such that, for all $x, y \in M$ and all $t \in (0, R^2)$,

$$h(t,x,y) \leq \frac{C_0}{\mu(B(x,\sqrt{t}))} \exp\left(-\frac{d(x,y)^2}{5t}\right). \tag{2.17}$$

4. For any integer k, there exists a constant C_k such that, for all $x, y \in M$, all $t \in (0, R^2)$,

$$|\partial_t^k h(t,x,y)| \leq \frac{C_k}{t^k \mu(B(x,\sqrt{t}))} \exp\left(-\frac{d(x,y)^2}{6t}\right). \tag{2.18}$$

5. There exists $c > 0$ such that, for all $x \in M$ and all $t \in (0, R^2)$,

$$h(t,x,x) \geq \frac{c}{\mu(B(x,\sqrt{t}))}. \tag{2.19}$$

These statements are essentially well-known, at least in the Riemannian case. The proof of "(2.14) implies (2.15)" in the general case need special care because no a priori asymptotic control of the volume of small balls is assumed here (in the Riemannian setting the volume of small balls is asymptotically Euclidean, see [7, 20]). A proof which works in the present generality is given in [2]. See also [40]. Note that (2.15) is a strong form of (2.2), see (2.3).

Inequality (2.19) does not seem to be in the literature except in [40]. See [9] for closely related statements. Since it plays an important part in our main argument, a proof is given in Section 3.3.

For the other implications, see e.g., [13, 20, 37, 38, 40, 43]. Let us point out that the fact that "(2.14) implies (2.17)" is not completely straightforward, even with a good knowledge of [11, 48]. As far as we know there are essentially two ways to prove this implication. One is to use (part of) Moser's iterative method together with Gaffney-Davies technique [11] as in [37, 43]. The other is to use the technique developed by A. Grigor'yan in [20]. It might also be possible to use the approach of [8]. Concerning (2.18), we note that it follows from (2.17) by a very general argument given in [13].

Remark As noted above, (2.15) implies the R-doubling property (2.2). It is not hard to check that R-doubling implies R'-doubling for any $R' = KR$, $K \in (1, +\infty)$. See (2.6). Using this fact and a covering argument, one also checks that the R-scale-invariant local Sobolev inequality (2.14) implies its R' analog for any $R' = KR$, $K \in (1, +\infty)$.

We will also need the following result.

Theorem 2.5 *Let $(\mathcal{E}, \mathcal{D}, L^2(M, d\mu))$ be a Dirichlet space such that the R-scale-invariant local Sobolev inequality (2.14) holds true for some fixed $R \in (0, +\infty]$. Then there exist positive reals A, a, ϵ_0 such that, for any ball B of radius $r(B) \leq \epsilon_0 R$ with $10B \neq M$, the least Dirichlet eigenvalue of L in B is bounded above and below by*

$$ar(B)^{-2} \leq \lambda_0(B) \leq Ar(B)^{-2}. \tag{2.20}$$

Proof The upper bound follows from Lemma 2.2. It is true for all $r < R$ (i.e., ϵ_0 is not needed here). The lower bound is more important for our purpose and we give a proof. Fix B with radius $r < R$. Using Jensen inequality for a function f supported in $\epsilon_0 B$, (2.14) implies

$$\int |f|^2 d\mu \leq \frac{S_0\, \mu(\epsilon_0 B)^{2/\nu}\, r(B)^2}{\mu(B)^{2/\nu}} \left(\int d\Gamma(f,f) + r(B)^{-2} \int |f|^2 d\mu\right).$$

In particular,

$$\left(1 - \frac{S_0 \mu(\epsilon_0 B)^{2/\nu}}{\mu(B)^{2/\nu}}\right) \int |f|^2 d\mu \leq \frac{S_0\, \mu(\epsilon_0 B)^{2/\nu}\, r(B)^2}{\mu(B)^{2/\nu}} \int d\Gamma(f,f).$$

By (2.16), this yields the desired result if ϵ_0 is chosen small enough. □

Remark Examples show that one can not dispense with the small constant ϵ_0 in Theorem 2.5.

The next result complements Theorem 2.4 by describing properties that are equivalent to (2.14).

Theorem 2.6 *Given a Dirichlet space $(\mathcal{E}, \mathcal{D}, L^2(M, d\mu))$ and $R \in (0, +\infty]$, the following properties are equivalent:*

1. *The R-scale-invariant local Sobolev inequality* (2.14) *holds true for some S_0 and $\nu > 2$.*
2. *Inequality* (2.15) *with $\nu > 2$ holds and there exists $C_0 > 0$ such that*

$$\forall\, x \in M, \ \ \forall\, t \in (0, R^2), \ \ h(t, x, x) \le \frac{C_0}{\mu(B(x, \sqrt{t}))}. \tag{2.21}$$

3. *There exist constants $a, \epsilon > 0$ such that for any ball B of radius less than ϵR, the relative Faber-Krahn inequality*

$$\text{for any open set } \Omega \subset B, \ \ \lambda_1(\Omega) \ge \frac{a}{r(B)^2} \left(\frac{\mu(B)}{\mu(\Omega)} \right)^{2/\nu} \tag{2.22}$$

   holds with $\nu > 2$.

In this theorem, the constant $\nu > 2$ is a fixed parameter. See [20, 37, 38, 39, 40].

As a corollary of Theorems 2.4, 2.6, we see that the on-diagonal upper bound (2.21) and R-doubling imply the Gaussian upper bounds (2.17) and (2.18). It is worth noting that the shortest and easiest path from (2.21) to (2.17) is described in [21]. The best way to get (2.18) is then to use the results of [13].

Similarly, the heat kernel upper bound (2.21) and R-doubling imply the Dirichlet eigenvalue estimate (2.20). For a direct proof of the lower bound, see, e.g., [20, 40].

Remark It is straightforward to check that property 2 in Theorem 2.6 extends from any fixed R to any $R' = KR$, $K \in (1, +\infty)$. Note that, in property 3 above, ϵ depends on R in general so that going from R to a larger R' does not produce any gain (a priori).

2.7 Poincaré inequality

Given $R \in (0, +\infty]$, we say that $(\mathcal{E}, \mathcal{D}, L^2(M, d\mu))$ satisfies a **R-scale-invariant Poincaré inequality** if there exists P_0 such that for any ball B of radius less than R,

$$\forall\, f \in \mathcal{D}, \quad \int_B |f - f_B|^2 d\mu \le P_0\, r(B)^2 \int_B d\Gamma(f, f) \tag{2.23}$$

where $f_B = \mu(B)^{-1} \int_B f d\mu$.

Remarks 1) It is important to observe that R-doubling and the R-Poincaré inequality (2.23) imply the R'-Poincaré inequality for any $R' = KR$, $K \in (1, +\infty)$. This is not obvious. To check it, use the technique of [10], a covering argument and the fact that any connected finite graph admits a Poincaré inequality with a constant depending only on the total number of vertices in the graph. Note that the number of vertices of each of the graphs involved is bounded uniformly because of the doubling property.

2) It is often easier to prove a weak form of the Poincaré inequality above where the left-hand side is replaced by

$$\int_{\tau B} |f - f_{\tau B}|^2 d\mu$$

for some fixed $\tau \in (0,1)$, e.g., $\tau = 1/2$. However, under the hypothesis that the space is R-doubling, this weak form of Poincaré inequality implies the strong form (2.23), a result due to D. Jerison [26]. See e.g., [40, 44].

The next result states that (2.2) and (2.23), together, characterize the Dirichlet spaces satisfying the parabolic Harnack inequality (2.11).

Theorem 2.7 ([19, 38, 44]) *Given $R \in (0,+\infty]$ and a Dirichlet space $(\mathcal{E}, \mathcal{D}, L^2(M, d\mu))$, the following properties are equivalent:*

1. *Inequalities (2.2) and (2.23) are satisfied.*
2. *The R-scale-invariant parabolic Harnack inequality (2.11) is satisfied.*
3. *The heat kernel $h(t,x,y)$ satisfies the two-sided Gaussian bound*
$$\frac{c_1}{\mu(B(x,\sqrt{t}))} \exp\left(-\frac{d(x,y)^2}{c_2 t}\right) \le h(t,x,y) \le \frac{C_1}{\mu(B(x,\sqrt{t}))} \exp\left(-\frac{d(x,y)^2}{C_2 t}\right)$$
for all $t \in (0,R^2)$ and $x,y \in M$.
4. *The R-doubling property (2.2) holds and for all $t \in (0,\epsilon_1 R^2)$, and all $x \in M$, the heat kernel satisfies*
$$h(t,x,x)\} \le \frac{C_3}{\mu(B(x,\sqrt{t}))} \quad \text{and} \quad \inf_{y\in B(x,\epsilon_2\sqrt{t})}\{h(t,x,y)\} \ge \frac{c_3}{\mu(B(x,\sqrt{t}))}$$
for some $C_3, c_3, \epsilon_1, \epsilon_2 > 0$.

We simply comment on these equivalences. "1 ⇒ 2" was proved independently by A. Grigory'an and the second author [19, 38]. This is, by far, the hardest part of the theorem. The paper [38] shows that "2 ⇒ 1", using an idea of Kusuoka and Stroock [28]. The implication "2 ⇒ 3" is not difficult thanks to Gaffney-Davies method for the upper bound. See e.g., [11, 37, 47, 48]. "3 ⇒ 4" is obvious. That "4 ⇒ 3" follows from Moser iteration and Gaffney-Davies type techniques for the upper bound. See, e.g., [37]. A different approach is described in [20]. The lower bound part follows from a well-known chaining argument. see, e.g., [36, pg 105]. Finally, "3 ⇒ 1" is not to hard to obtain. See [36, 38, 40]. It is worth mentioning that one can prove directly that "3 ⇒ 2" (without passing through 1). The details of this implication can be found in [16, Sect. 3]. The setting in [16] is different but the argument is easily adapted to the present situation. This is noteworthy because the implication "3 ⇒ 2", together with the technique presented in this paper, offers an alternative route to show that 1 implies 2. We believe that this proof of "1 ⇒ 2" is of some interest. See Section 4.2.

3 Elliptic and parabolic Harnack inequalities

3.1 The main results

The main results of the present paper are stated in the following theorem and corollaries. These results are new even in the case of Riemannian manifolds equipped with their canonical Dirichlet space structure corresponding to the minimal closure of the form

$$(\mathcal{E}(f,f) = \int_M |\nabla f|^2 dv, \mathcal{C}_0^\infty(M)).$$

In the statements below, $(\mathcal{E}, \mathcal{D}, L^2(M, d\mu))$ is a Dirichlet space as in Section 2 and $R \in (0,+\infty]$ is fixed.

Theorem 3.1 *Assume that the R-scale-invariant local Sobolev inequality* (2.14) *is satisfied. Assume also that the elliptic Hölder regularity estimate* (2.12) *is satisfied. Then the R-scale-invariant parabolic Harnack inequality* (2.11) *is satisfied and so is the parabolic Hölder regularity estimate* (2.13).

Corollary 3.2 *Assume that the R-scale-invariant local Sobolev inequality* (2.14) *is satisfied. Assume also that the R-scale-invariant elliptic Harnack* (2.10) *is satisfied. Then the R-scale-invariant parabolic Harnack inequality* (2.11) *is satisfied and so is the parabolic Hölder regularity estimate* (2.13).

Corollary 3.3 *Assume that the R-scale-invariant local Sobolev inequality* (2.14) *is satisfied. Then the R-scale-invariant elliptic Harnack* (2.10), *R-scale-invariant Hölder regularity estimate* (2.12), *and the R-scale-invariant Poincaré inequality* (2.23) *are equivalent properties.*

Corollary 3.4 *The following properties are equivalent to the equivalent properties 1-4 of Theorem 2.7:*

5. *The R-scale-invariant local Sobolev inequality* (2.14) *and either the R-scale-invariant elliptic Harnack* (2.10) *or the Hölder continuity estimate* (2.12) *are satisfied.*
6. *The R-doubling property* (2.2), *the heat kernel upper bound* (2.21), *and either the R-scale-invariant elliptic Harnack inequality* (2.10) *or the Hölder continuity estimate* (2.12) *are satisfied.*

In all these statements, R can be either finite or $+\infty$. If R is finite, its exact value is irrelevant, to some extend, because if any of these properties (hypotheses or conclusions) is satisfied for a given finite R, it is also satisfied for all finite R', with constants depending on R'.

These results are of theoretical value. For instance, it is intriguing that the "gap" between the Sobolev inequality (2.14) and the conjunction of the doubling property and Poincaré inequality (2.2), (2.23) can be bridged using the elliptic Hölder continuity estimate (2.12). They are also of a certain practical interest. As we shall see, the new parts of the proof of Theorem 3.1 are rather direct and simple. In some sense, they yield a reasonable route to the parabolic Harnack inequality (2.11) passing through the simpler elliptic case. See the remarks following Theorem 2.7 above as well as Section 4.2 below. This route to the parabolic Harnack inequality seems especially valuable in the setting of analysis on graphs which is not covered by the present strictly local Dirichlet space framework. In fact, the results above originated from our desire to overcome some of the difficulties that appear in the case of graphs. See [14, 1]. This will be developed elsewhere.

3.2 The heart of the proof

To isolate the part of the proof of Theorem 3.1 that is new, we formulate the following result.

Proposition 3.5 *Fix $R \in (0, +\infty]$. Assume that no ball of radius $10R$ covers M. Assume also that the R-doubling property* (2.2), *the heat kernel upper bound* (2.17), *and the Hölder continuity estimate* (2.12) *are satisfied. Then there exist $\epsilon_i > 0$, $i = 1, 2, 3$, such that the heat kernel $h(t, x, y)$ satisfies the lower bound*

$$\inf_{y \in B(x, \epsilon_1 \sqrt{t})} \{h(t, x, y)\} \geq \frac{\epsilon_2}{\mu(B(x, \sqrt{t}))}$$

for all $t \in (0, \epsilon_3 R^2)$, $x \in M$.

Once this proposition as been proved, it follows by well-established arguments that the heat kernel satisfies the two-sided Gaussian bound of Theorem 2.7(3) for $t \in (0, \epsilon_5 R)$ for some $\epsilon_5 > 0$. Note that if M is not compact, the restriction that no ball of radius $10R$ covers M is void. If M is compact, we can work with R small enough so that this condition is satisfied.

Next, the $(\epsilon_5 R)$-parabolic Harnack inequality (2.11) can be obtained following the line of reasoning of [16, Sect. 3]. As mentioned after Theorem 2.7, this automatically imply the R-version of the same inequality. Finally, as the Sobolev inequality (2.14) implies (2.2) and (2.17), we have a complete proof of Theorem 3.1, assuming we can prove Proposition 3.5.

The idea of the proof of Proposition 3.5 in the case where $R = +\infty$ is the following. We want to show that

$$h(t,x,y) \geq \frac{\epsilon_2}{\mu(B(x,\sqrt{t}))}$$

for all $x \in M$, all $t > 0$ and all $y \in B(x, \epsilon_1\sqrt{t})$, for some $\epsilon_1, \epsilon_2 > 0$. The Gaussian upper bound (2.17) and R-doubling imply the on diagonal lower bound

$$\forall\, x \in M,\ \forall\, t > 0,\ \ h(t,x,x) \geq \frac{c_1}{V(x,\sqrt{t})}.$$

Hence, it suffices to show that

$$|h(t,x,y) - h(t,x,x)| \leq C\left(\frac{d(x,y)}{\sqrt{t}}\right)^{\alpha} \frac{1}{\mu(B(x,\sqrt{t}))}.$$

Assuming transience, that is the existence of the Green function

$$G(x,y) = \int_0^\infty h(s,x,y)ds < +\infty$$

for all $x \neq y$, the inequality above can be proved by writing

$$\begin{aligned} h(t,x,y) - h(t,x,z) &= L_y^{-1}L_y(h(t,x,y) - L_z^{-1}L_z h(t,x,z) \\ &= L_y^{-1}\partial_t h(t,x,y) - L_z^{-1}\partial_t h(t,x,z) \\ &= \int_M [G(y,\zeta) - G(z,\zeta)]\partial_t h(t,x,\zeta)d\mu(\zeta). \end{aligned}$$

The point of this formula is that, under our hypotheses, one can : 1) estimate $|\partial_t h(t,x,\zeta)|$ and 2) use the elliptic Hölder continuity estimate to bound $|G(y,\zeta) - G(z,\zeta)|$ since the Green function G satisfies $L_\zeta G(y,\zeta) = 0$ for $\zeta \neq y$. As far as we can say, to make this line of reasoning work, transience is not quite enough but a uniform volume estimate of the form $\mu(B(x,s)) \geq c(s/r)^\gamma \mu(B(x,r))$ with $\gamma > 2$ suffices. However, this or even transience are very **unnatural** hypotheses in the present problem. It turns out that one can modify the above argument to cover the general case. To achieve this we will work locally using the Dirichlet heat kernel on various balls of well chosen radii. As it turns out, our proof is closely related to techniques used in analysis on fractals, e.g., [4, 5].

3.3 Proof of Proposition 3.5

Throughout this section and the next we assume that $(\mathcal{E}, \mathcal{D}, L^2(M, d\mu))$ satisfies the hypotheses of Proposition 3.5, that is, for a certain fixed $R \in (0, +\infty]$, the R-doubling property (2.2) holds, the Gaussian heat kernel upper bound (2.17) holds, and the elliptic Hölder continuity estimate (2.12) holds. Note that each of these three hypotheses involves the parameter R. We also assume that no balls of radius $10R$ covers M.

We start with the following observation: A simple consequence of (2.2) and (2.17) is that there exists $c_1 > 0$ such that

$$\forall\, t \in (0, R^2),\ \forall\, z \in M,\ \ h(t,z,z) \geq \frac{c_1}{\mu(B(z,\sqrt{t}))}. \tag{3.1}$$

This is stated in Theorem 2.4(5). Since this is an important step in our argument, we give a complete proof. Write

$$
\begin{aligned}
h(t,x,x) &= \int_M |h(t/2,x,y)|^2 d\mu(y) \geq \int_{B(x,\sqrt{T})} |h(t/2,x,y)|^2 d\mu(y) \\
&\geq \frac{1}{\mu(B(x,\sqrt{T}))}\left(\int_{B(x,\sqrt{T})} h(t/2,x,y)d\mu(y)\right)^2 \\
&= \frac{1}{\mu(B(x,\sqrt{T}))}\left(\int_M h(t/2,x,y)d\mu(y) - \int_{M\setminus B(x,\sqrt{T})} h(t/2,x,y)d\mu(y)\right)^2.
\end{aligned}
$$

Now, stochastic completeness, i.e., $\int_M h(t/2,x,\cdot)d\mu = 1$ (see Theorem 2.1) yields

$$
h(t,x,x) \geq \frac{1}{\mu(B(x,\sqrt{T}))}\left(1 - \int_{M\setminus B(x,\sqrt{T})} h(t/2,x,y)d\mu(y)\right)^2. \tag{3.2}
$$

Here, T can be thought of as a large multiple of t to be chosen later. To finish the proof, we need the following elementary lemma.

Lemma 3.6 *Properties* (2.2) *and* (2.17) *imply that there exist two positive reals K, c such that, for all $s \in (0, R^2)$, all $T > s$ and all $x \in M$*

$$
\int_{M\setminus B(x,\sqrt{T})} h(s,x,y)d\mu(y) \leq K e^{-cT/s}.
$$

Proof Write

$$
\begin{aligned}
\int_{M\setminus B(x,\sqrt{T})} h(s,x,y)d\mu(y) &= \sum_1^\infty \int_{2^{i-1}\sqrt{T}\leq d(x,y)<2^i\sqrt{T}} h(s,x,y)d\mu(y) \\
&\leq C\sum_1^\infty \frac{\mu(B(x,2^i\sqrt{T}))}{\mu(B(x,\sqrt{s}))}\exp\left(-\frac{4^iT}{20s}\right) \\
&\leq C'\sum_1^\infty \exp\left(D\frac{2^i\sqrt{T}}{\sqrt{s}} - \frac{4^iT}{20s}\right) \leq C''\exp\left(-\frac{T}{10s}\right).
\end{aligned}
$$

Here we have used (2.17) and (2.6). □

Now, chose $s = t/2$, $T = ks$ with k large enough so that

$$
\int_{M\setminus B(x,\sqrt{T})} h(t/2,x,y)d\mu(y) \leq \frac{1}{2}.
$$

This, together with (3.2), proves (3.1).

Recall that we want to prove there exist $\epsilon_1, \epsilon_2, \epsilon_3 > 0$ such that

$$
\forall\, x \in M,\ \ \forall\, t \in (0, \epsilon_3 R^2),\ \ \forall\, y \in B(x, \epsilon_1\sqrt{t}),\quad h(t,x,y) \geq \frac{\epsilon_2}{\mu(B(x,\sqrt{t}))}. \tag{3.3}
$$

Fix $x \in M$, $t \in (0, R^2)$. Let $B = B(x,\rho)$ with $\rho \geq \sqrt{t}$ to be chosen later and consider the Dirichlet heat kernel $h^B(t,z,y)$ in B. As

$$
h(t,x,y) \geq h^B(t,x,y)
$$

for all $y \in B$, it suffices to show that for all $x \in M$ and all $t \in (0, \epsilon_3 R^2)$,

$$\forall\, y \in B(x, \epsilon_1 \sqrt{t}), \quad h^B(t, x, y) \geq \frac{\epsilon_2}{\mu(B(x, \sqrt{t}))}. \tag{3.4}$$

The first step is to transfer the lower bound (3.1) to the Dirichlet heat kernel.

Lemma 3.7 *There exists $c > 0$ such that for any A_0 large enough, any $x \in M$, any $t \in (0, R^2)$, and any $\rho \geq A_0 \sqrt{t}$*

$$h^B(t, x, x) \geq \frac{c}{V}$$

where $B = B(x, \rho)$ as above and $V = \mu(B(x, \sqrt{t}))$.

Proof Consider the Hunt process X associated with $(\mathcal{E}, \mathcal{D}, L^2(M, d\mu))$. See [18, Ch 4,7]. Let τ be the first exit time from B. Then the Dirichlet heat kernel can be expressed by the Dynkin-Hunt formula

$$h^B(t, x, y) = h(t, x, y) - E^x \left[h(t - \tau, X_\tau, y) \mathbf{1}_{\{\tau \leq t\}} \right].$$

Thus, by (3.1) and (2.17),

$$\begin{aligned} h^B(t, x, x) &= h(t, x, x) - E^x \left[h(t - \tau, X_\tau, x) \mathbf{1}_{\{\tau \leq t\}} \right] \\ &\geq \frac{c_1}{V} - \sup_{0<s<t} \frac{C}{\mu(B(x, \sqrt{s}))} e^{-\rho^2/5s} \\ &\geq \left(c_1 - C \frac{V}{\mu(B(x, \sqrt{s}))} e^{-A_0^2 t/5s} \right) \frac{1}{V} \\ &\geq \left(c_1 - C D_0 (t/s)^{\nu/2} e^{-A_0^2 t/5s} \right) \frac{1}{V} \\ &\geq \left(c_1 - C D_0 (5\nu/2A_0^2)^{\nu/2} \right) \frac{1}{V}. \end{aligned}$$

Here we have use (2.3) with $\nu = \log_2(D_0)$. Clearly, we can choose A_0 large enough so that

$$c = \left(c_1 - C D_0 (5\nu/2A_0^2)^{\nu/2} \right) = c_1/2 > 0.$$

□

Let us now state a crucial technical result and show how (3.4), hence Proposition 3.5, follows from it.

Lemma 3.8 *For any $\sigma > 0$ and any $A \geq 1$ there exist two positive reals $C_{\sigma,A}$ and ϵ_A such that for all $x \in M$, $t \in (0, \epsilon_A R^2)$, and $y \in B$,*

$$|h^B(t, x, y) - h^B(t, x, x)| \leq \left[\sigma + C_{\sigma,A} \left(\frac{d(x, y)}{\sqrt{t}} \right)^\alpha \right] \frac{1}{V}$$

where $B = B(x, \rho)$ with $\rho = A\sqrt{t}$, $V = \mu(B(x, \sqrt{t}))$ as above, and α is the Hölder exponent in (2.12).

The proof of this lemma depends on a number of technical estimates. Before embarking on this proof, observe that (3.4), hence Proposition 3.5, follows easily from Lemmas 3.7 and 3.8. Indeed, let A_0, c be the constants given by Lemma 3.7. Let $\sigma = c/2$ and $A = A_0$ so that the conclusion of Lemma 3.8 applies. Then, for $y \in B$, and $t \in (\epsilon_A R^2)$,

$$\begin{aligned} h(t, x, y) &\geq h^B(t, x, y) \\ &\geq h^B(t, x, x) - \left[\frac{c}{2} + C \left(\frac{d(x, y)}{\sqrt{t}} \right)^\alpha \right] \frac{1}{V} \\ &\geq \frac{c}{V} - \left[\frac{c}{2} + C \left(\frac{d(x, y)}{\sqrt{t}} \right)^\alpha \right] \frac{1}{V} \\ &\geq \left[\frac{c}{2} - C \left(\frac{d(x, y)}{\sqrt{t}} \right)^\alpha \right] \frac{1}{V} \end{aligned}$$

Thus, for $\epsilon_1 > 0$ small enough and $y \in B(x, \epsilon_1\sqrt{t})$, we have

$$h(t,x,y) \geq \frac{c}{4V}$$

as desired. We are left with the task of tackling Lemma 3.8.

3.4 Proof of Lemma 3.8

In this section we assume that the hypotheses of Proposition 3.5 are satisfied. We fix $x \in M$, we let $A = A_0$ be the large constant given by Lemma 3.7 and set $\rho = A\sqrt{t}$ with $t \in (0, \epsilon R^2)$ and $\epsilon \in (0, A^{-2})$ to be chosen later. We let

$$B = B(x,\rho) \quad \text{and} \quad V = \mu(B(x,\sqrt{t})).$$

Note that, by construction and our hypotheses, the ball $10B$ does not cover M. We will need some upper estimates on $h^B(s,z,y)$ and its time derivative $\partial_s h^B(s,z,y)$.

Lemma 3.9 *Let $B = B(x,\rho)$ with $\rho = A\sqrt{t}$ and $V = \mu(B(x,\sqrt{t}))$ as above. Assume that the Gaussian upper bound* (2.17) *and the R-doubling property* (2.2) *are satisfied. Then the following estimates hold:*

1. *There exists C_1 such that*

$$\forall\, s \in (0,R^2),\ \forall\, z,y \in B, \quad h^B(s,y,z) \leq \frac{C_1}{\mu(B(y,\sqrt{s}))} \exp\left(-\frac{d(z,y)^2}{5s}\right). \tag{3.5}$$

2. *There exists C_2 such that*

$$\forall\, z,y \in B, \quad |\partial_t h^B(t,z,y)| \leq \frac{C_2 A^\nu}{tV} \tag{3.6}$$

   where $\nu = \log_2(D_0)$, with D_0 the constant appearing in (2.2).

3. *There exists $\epsilon, a_1 > 0$, and, for any $0 < \theta < 1$, there exists C_θ such that, for any $t \in (0, \epsilon R^2)$,*

$$\forall\, s \in (\theta t, +\infty)\ \forall\, z,y \in B, \quad h^B(s,y,z) \leq \frac{C_\theta A^\nu}{V} \exp\left(-\frac{a_1 s}{A^2 t}\right). \tag{3.7}$$

Proof The first bound is obvious since $h^B \leq h$. For the second inequality, write

$$\begin{aligned}
|\partial_t h^B(t,z,y)| &= \left|\int \partial_t h^B(3t/4,z,\xi) h^B(t/4,\xi,y)\,d\mu(\xi)\right| \\
&\leq \|L_y^B h^B(3t/4,z,\cdot)\|_2 \|h^B(t/4,\cdot,y)\|_2 \\
&\leq \frac{1}{t}\|h^B(t/4,z,\cdot)\|_2 \|h^B(t/4,\cdot,y)\|_2 \\
&\leq \frac{1}{t}\sqrt{h^B(t/2,z,z)h^B(t/2,y,y)} \leq \frac{C_2 A^\nu}{tV},
\end{aligned}$$

where we used the semigroup property, the spectral theorem, (3.5) and (2.4).

For the third inequality, recall first that the $L^2 - L^\infty$-operator norm of H_s^B is

$$\|H_s\|_{2\to\infty} = \sup_{z\in B} \|h^B(s/2,z,\cdot)\|_2.$$

Then write

$$\begin{aligned}\sup_{z,y\in B}\{h^B(s,z,y)\} &= \sup_{z\in B}\{\|h^B(s/2,z,\cdot)\|_2^2\} = \|H^B_{s/2}\|^2_{2\to\infty}\\ &\le \|H^B_{s/2-\theta t/4}\|^2_{2\to 2}\|H^B_{\theta t/4}\|^2_{2\to\infty}\\ &\le \frac{C_\theta A^\nu}{V}\exp\left(-\frac{as}{2\rho^2}\right)\end{aligned}$$

where a is given by the lower bound in Theorem 2.5. This gives (3.7) with $a_1 = a/2$. $\square$

Remarks 1) The constants C_1, C_2, C_θ and a_1 do not depend on x, t, A.

2) The need to restrict t to the range $(0, \epsilon R^2)$ with ϵ small enough in Lemma 3.9 comes from the use of Theorem 2.5.

From now on, we assume that $t < \epsilon R^2$ with $0 < \epsilon < A^{-2}$ given by Lemma 3.9. The proof of Lemma 3.8 starts as follows. Let $G^B(y,z) = \int_0^\infty h^B(s,y,z)ds$ be the Green function with Dirichlet boundary condition in B. Write

$$\begin{aligned}h^B(t,x,y) - h^B(t,x,z) &= (L_y^B)^{-1}L_y^B(h(t,x,y) - (L_z^B)^{-1}L_z^B h^B(t,x,z)\\ &= \int_B [G^B(y,\zeta) - G(z,\zeta)]\partial_t h^B(t,x,\zeta)d\mu(\zeta).\end{aligned}$$

In particular, for $z = x$,

$$|h^B(t,x,y) - h^B(t,x,x)| \le \int_B |G^B(y,\zeta) - G^B(x,\zeta)||\partial_t h^B(t,x,\zeta)|d\mu(\zeta). \tag{3.8}$$

For any $\eta \in (0,1)$, write the right-hand side of (3.8) as the sum of three terms I_1, I_2, J corresponding to integration over the sets

$$W_1 = \{\zeta \in B : d(x,\zeta) \le \eta\sqrt{t}\},\ \ W_2 = \{\zeta \in B : d(y,\zeta) \le \eta\sqrt{t}\}$$

for I_1, I_2 and

$$W = \{\zeta \in B : d(x,\zeta) \ge \eta\sqrt{t} \text{ and } d(y,\zeta) \ge \eta\sqrt{t}\}$$

for J.

The next lemma bounds I_1, I_2.

Lemma 3.10 *For any $A, \tau > 0$, there exists $\eta_{\tau,A} > 0$ small enough so that, for all $y \in B$,*

$$I_i \le \frac{\tau}{V}, \quad i = 1, 2.$$

Proof We treat I_1. The same argument work for I_2 (using (2.4)). Write

$$I_1 \le \left(\sup_{\zeta,\xi}\{|\partial_t h^B(t,\zeta,\xi)|\}\right)\int_{W_1}[G^B(x,\zeta) + G^B(y,\zeta)]d\mu(\zeta).$$

By (3.6),

$$\sup_{\zeta,\xi}\{|\partial_t h^B(t,\zeta,\xi)|\} \le \frac{C_2 A^\nu}{tV}. \tag{3.9}$$

To estimate $\int_{W_1} G^B(x,\zeta)d\mu(\zeta)$ write

$$\int_{W_1} G^B(x,\zeta)d\mu(\zeta) = \int_{W_1}\left(\int_0^{\theta t} h^B(s,x,\zeta)ds + \int_{\theta t}^\infty h^B(s,x,\zeta)ds\right)d\mu(\zeta).$$

Then use (3.7) and (2.16) to obtain

$$\begin{aligned}\int_{W_1} G^B(x,\zeta)d\mu(\zeta) &\leq \theta t + \frac{C_\theta A^\nu \mu(W_1)}{V}\int_{\theta t}^\infty e^{-a_1 s/A^2 t}ds \\ &\leq \theta t + \frac{C_\theta A^{2+\nu} t\mu(W_1)e^{-a_1\theta/A^2}}{a_1 V} \\ &\leq \left(\theta + \frac{C_\theta A^{2+\nu}\eta^\gamma}{a_1\,\beta}\right) t \qquad (3.10)\end{aligned}$$

where, for the last inequality, we have used that the R-doubling inequality (2.2) and the hypothesis that no ball of radius $10R$ covers M imply there exist $\beta, \gamma > 0$ such that (2.16) holds true.

By (3.9), (3.10) we get

$$I_1 \leq C_2 A^\nu \left(\theta + \frac{C_\theta A^{2+\nu}\eta^\gamma}{a_1\,\beta}\right)\frac{1}{V}.$$

For any $\tau > 0$, pick

$$\theta = \frac{\tau}{2C_2 A^\nu}$$

and $\eta = \eta_\tau$ so that

$$\frac{C_\theta A^{2+\nu}\eta^\gamma}{a_1\,\beta} = \frac{\tau}{2C_2 A^\nu}.$$

This yields the inequality claimed in Lemma 3.10. □

We now focus on J and start with a Hölder continuity estimate for the Green function G^B in an appropriate subset of B.

Lemma 3.11 *For any $A, \tau > 0$, let $\eta = \eta_{\tau,A}$ be given by Lemma 3.10. Let*

$$W = \{\zeta \in B : d(x,\zeta) \geq \eta\sqrt{t} \text{ and } d(y,\zeta) \geq \eta\sqrt{t}\}.$$

Then there exists $C_{\tau,A}$ such that the Green function G^B satisfies

$$|G^B(x,\zeta) - G^B(y,\zeta)| \leq C_{\tau,A}\left(\frac{d(x,y}{\sqrt{t}}\right)^\alpha \frac{t}{V}$$

for all $y \in B$ and all $\zeta \in W$. Here α is the Hölder regularity exponent given by (2.12).

Proof Let us start with an estimate of $G^B(z,\zeta)$ when $d(z,\zeta) \geq \eta\sqrt{t}/2$, $z \in B$. In this case, we have

$$\begin{aligned}G^B(z,\zeta) &= \int_0^t h^B(s,z,\zeta)ds + \int_t^\infty h^B(s,z,\zeta)ds \\ &\leq \frac{C_1}{V}\int_0^t \frac{V}{\mu(B(z,\sqrt{s}))}e^{-\eta^2 t/5s}ds + \frac{C_2}{V}\int_t^\infty e^{-a_1 s/(A^2 t)}ds \\ &\leq \left(\frac{C_1' A^\nu}{\eta^\nu} + C_2' A^2\right)\frac{t}{V}. \qquad (3.11)\end{aligned}$$

Here $\nu = \log_2 D_0$ again and we have used (2.4) to bound $V/\mu(B(z,\sqrt{t}))$ from above.

For $\zeta \in W$, the function $z \mapsto G^B(z,\zeta)$ is harmonic in $B(x,\eta\sqrt{t})$. Thus, by hypothesis, for any $y \in B(x,\eta\sqrt{t}/4)$,

$$\begin{aligned}|G^B(x,\zeta) - G^B(y,\zeta)| &\leq C\left(\frac{d(x,y)}{\eta\sqrt{t}}\right)^\alpha \sup_{B(x,\eta\sqrt{t}/2)}\{G^B(\cdot,\zeta)\} \\ &\leq C_{\tau,A}\left(\frac{d(x,y)}{\sqrt{t}}\right)^\alpha \frac{t}{V}.\end{aligned}$$

Now, if $y \notin B(x, \eta\sqrt{t}/4)$, (3.11) yields

$$\begin{aligned} |G^B(x,\zeta) - G^B(y,\zeta)| &\leq G^B(x,\zeta) + G^B(y,\zeta) \\ &\leq C_{\tau,A}\frac{t}{V}. \end{aligned}$$

In both cases, the desired estimate follows. □

Lemma 3.12 *For any $A, \tau > 0$, let $\eta = \eta_\tau$ be given by Lemma* 3.10*. Let*

$$W = \{\zeta \in B : d(x,\zeta) \geq \eta\sqrt{t} \text{ and } d(y,\zeta) \geq \eta\sqrt{t}\}.$$

Then there exist $C_{\tau,A}$ such that, for any $y \in B$,

$$J = \int_W |G^B(x,\zeta) - G^B(y,\zeta)| h^B(t,x,\zeta) d\mu(\zeta) \leq C_{\tau,A}\left(\frac{d(x,y}{\sqrt{t}}\right)^\alpha \frac{1}{V}$$

where α is the Hölder exponent in (2.12)*.*

Proof This easily follows from Lemma 3.11, (3.7) and (2.2). □

We can now put the pieces together and finish the proof of Lemma 3.8. Fix $\sigma > 0$. Apply (3.8) and Lemmas 3.10, 3.12 with $\tau = \sigma/2$. Then,

$$\begin{aligned} |h^B(t,x,y) - h^B(t,x,x)| &\leq I_1 + I_2 + J \\ &\leq \left(\sigma + C_\sigma\left(\frac{d(x,y}{\sqrt{t}}\right)^\alpha\right)\frac{1}{V} \end{aligned}$$

as desired. This finishes the proof of Lemma 3.8.

4 Further comments and results

4.1 Elliptic Harnack inequality in $\mathbb{R}^m \times M$

Consider the following problem. Let $(\mathcal{E}, \mathcal{D}, L^2(M, d\mu))$ be a Dirichlet space as in Section 2 with corresponding heat kernel $h(t,x,y)$ and infinitesimal generator $-L$.

Consider the space $M_{(m)} = \mathbb{R}^m \times M$ equipped with the measure $d\mu_{(m)} = d\lambda \times d\mu$ where $d\lambda$ denotes the Lebesgue measure on $\mathbb{R}^m$. Let $\Delta = -\sum_1^m \partial_i^2$ be the canonical Laplace operator on the Euclidean space $\mathbb{R}^m$. When does an elliptic Harnack inequality hold true for non-negative solutions of the elliptic equation

$$(\Delta + L)u = 0?$$

To be precise, fix $R \in (0, +\infty]$. We say that a R-scale-invariant elliptic Harnack inequality holds for $L_{(m)} = \Delta + L$ if there exists a constant C such that, for any $s \in \mathbb{R}^m, x \in M, r \in (0, R)$ and any non-negative solution u of $L_{(m)}u = 0$ in a box $K(s,x,r) = \mathbb{B}(s,r) \times B(x,r)$, the inequality

$$\sup_{K(s,x,r/2)}\{u\} \leq C \inf_{K(s,x,r/2)}\{u\}$$

holds true. Here $\mathbb{B}(s,r)$ denote the Euclidean ball of radius r around $s \in \mathbb{R}^m$. It is not hard to see that this is equivalent to the R-scale-invariant elliptic Harnack inequality (2.10) for the natural Dirichlet space on $M_{(m)}$ associated with $L_{(m)}$. The only difference is that we have used the boxes $K(s,x,r)$ above instead of the intrinsic balls $B((s,x),r)$ associated with $L_{(m)}$. But, obviously, these intrinsic balls and the boxes $K(s,x,r)$ are comparable in the sense that there are $C, c > 0$ such that $K(s,x,cr) \subset B((s,x),r) \subset K(s,x,Cr)$.

In our general setting, the reader might wonder what is the exact meaning of $L_{(m)} = \Delta + L$. One easy way to deal with this question is to define $L_{(m)}$ to be minus the infinitesimal generator of the product semigroup $H_t^{(m)} = H_t^{\mathbb{R}^m} \otimes H_t$ on $\mathbb{R}^m \times M$ where $H_t^{\mathbb{R}^m} = e^{t(\sum_1^m \partial_i^2)}$ is the Euclidean heat diffusion semigroup. This takes care at ounce of the problem of defining the Dirichlet space $(\mathcal{E}_{(m)}, \mathcal{D}_{(m)})$ on $\mathbb{R}^m \times M$ associated with $L_{(m)}$. Note that $L_{(0)} = L$.

Theorem 4.1 *Fix $R \in (0, +\infty]$ and a positive integer m. Then the operator $L_{(m)}$ satisfies a R-scale-invariant elliptic Harnack inequality if and only if L satisfies a R-scale-invariant parabolic Harnack inequality.*

Proof Assume first that L satisfies a parabolic Harnack inequality. By Theorem 2.7, (2.2) and (2.23) hold true on M. This implies that the same inequalities (up to changes in the constants) hold on $\mathbb{R}^m \times M$ equipped with the Dirichlet space structure $(\mathcal{E}_{(m)}, \mathcal{D}_{(m)})$ defined above. Thus a R-scale-invariant elliptic (in fact, parabolic) Harnack inequality holds for $L_{(m)}$. $\square$

Let us now assume that $L_{(1)}$ satisfies a R-scale-invariant elliptic Harnack inequality. Let P_t denote the (Poisson) semigroup with infinitesimal generator $-\sqrt{L}$. This semigroup can be obtained from H_t by the subordination formula

$$P_t = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}} H_{t^2/4u} du.$$

We let $p(t, x, y)$ be the kernel of P_t which can be obtained from $h(t, x, y)$ by the same formula as above.

Lemma 4.2 *Assume that $L_{(1)}$ satisfies a R-scale-invariant elliptic Harnack inequality. Then, for all $x \in M$ and all $0 < t < R/2$,*

$$p(t, x, x) \le \frac{C}{\mu(B(x, t))}$$

where C is the constant appearing in the postulated Harnack inequality.

Proof As $(t, y) \mapsto u(t, y) = p(t, x, y)$ is a solution of $L_{(1)} u = 0$ in $(0, \infty) \times M$, we have

$$\forall\, x \in M,\ \forall\, 0 < t < R/2,\ \ \forall\, y \in B(x, t), \quad p(t, x, x) \le C p(2t, x, y). \tag{4.12}$$

Integrating over $B(x, t)$ gives

$$p(t, x, x)\mu(B(x, t)) \le C \int_{B(x,t)} p(2t, x, y) d\mu(y) \le C,$$

as desired. $\square$

Lemma 4.3 *Assume that $L_{(1)}$ satisfies a R-scale-invariant elliptic Harnack inequality. Then, there exist C_1 and $N > 0$ such that such that, for all $x \in M$, $0 < t < R/2$ and $0 < s \le t$,*

$$p(s, x, x) \le C_1 \left(\frac{t}{s}\right)^N \frac{1}{\mu(B(x, t))}.$$

Proof. Fix x and $0 < s \le t < R$. Let k be the integer such that $2^{k-1} \le t/s < 2^k$. Let $t_i = 2^i s$, $i = 0, \dots, k$. Applying (4.12) with $y = x$ and $t = t_i$, $i = 0, \dots, k$, we get

$$\log\left(\frac{p(t_i, x, x)}{p(t_{i+1}, x, x)}\right) \le \log C.$$

This yields

$$\log\left(\frac{p(s,x,x)}{p(2^k s,x,x)}\right) \le (\log C)(1+k).$$

As $p(t,x,x)$ is a non-increasing function of t and $k \le 1 + \log_2(t/s)$, we get

$$p(s,x,x) \le C^2 (t/s)^N p(t,x,x)$$

with $N = \log_2 C$ where $\log_2(u) = \log u / \log 2$. By Lemma 4.2, the desired result follows. □

Lemma 4.4 *Assume that $L_{(1)}$ satisfies a R-scale-invariant elliptic Harnack inequality. Then, there exist S_0 and $\nu > 2$ such that the Dirichlet space $(\mathcal{E}, \mathcal{D})$ satisfies a R-scale-invariant local Sobolev inequality as in* (2.14)*.*

Fix a ball $B = B(x,t)$, $0 < t < R/2$ and consider the operator L with Dirichlet boundary condition in B (of course, this is an abuse of language). Let $p^B(t,x,y)$ the kernel of the Poisson semigroup with Dirichlet boundary condition in B. We have

$$\forall\, t > 0,\ \ \forall\, x,y \in M, \quad p^B(t,x,y) \le p(t,x,y).$$

By a result of N. Varopoulos (see e.g., [48, II.4.2]) the last inequality and Lemmas 3.9, 4.3 together yield the desired local Sobolev inequality on B with $\nu = \max\{3, N\}$ where N is as in Lemma 4.3. This gives a $(R/2)$-scale-invariant local Sobolev inequality. By the remark following Theorem 2.4, a R-scale-invariant local Sobolev inequality follows as well.

By Theorem 3.1 and Lemma 4.4, it is now clear that a scale-invariant elliptic Harnack inequality for $L_{(1)}$ implies a corresponding parabolic Harnack inequality for L. This finishes the proof of Theorem 4.1. □

It might be worth emphasizing some immediate corollaries.

Corollary 4.5 *Assume that $L_{(1)}$ satisfies a R-scale-invariant elliptic Harnack inequality. Then the Dirichlet space $(\mathcal{E}, \mathcal{D}, L^2(M, d\mu)$ is R-doubling*

This follows from Lemma 4.4 and Theorem 2.4. Note that the proof is rather indirect. Note also that **it is not true** that a R-scale-invariant elliptic Harnack inequality for L implies R-doubling. See the examples from [4, 15].

Corollary 4.6 *Fix $0 < R \le \infty$. The following are equivalent properties:*

1. *There exists a positive integer m such that $L_{(m)}$ satisfies the R-scale-invariant elliptic Harnack inequality* (2.10)*.*
2. *For any $m = 0, 1, 2, \dots,$ $L_{(m)}$ satisfies the R-scale-invariant parabolic Harnack inequality* (2.11)*.*
3. *The original operator $L = L_{(0)}$ satisfies the R-scale-invariant parabolic Harnack inequality* (2.11)*.*
4. *The Dirichlet space $(\mathcal{E}, \mathcal{D}, L^2(M, d\mu))$ is R-doubling and satisfies a R-scale-invariant Poincaré inequality.*

4.2 A new proof of the parabolic Harnack inequality

This section outline a new proof of the parabolic Harnack inequality (2.11), under the hypothesis that $(\mathcal{E}, \mathcal{D}, L^2(M d\mu))$ satisfies the R-doubling inequality (2.2) and the R-scale-invariant Poincaré inequality (2.23). From a technical point of view, it is worth noting that, for the purpose of the proof to be given below, (2.23) can be replaced by its weaker form

$$\int_{\tau B} |f - f_{\tau B}|^2 d\mu \leq P_0\, r(B)^2 \int_B \Gamma(f,f), \tag{4.13}$$

for some fixed $\tau \in (0,1)$, e.g., $\tau = 1/2$.

The first step of the proof is to obtain the elliptic Harnack inequality (2.10) or the elliptic Hölder continuity estimate (2.12). This can be done using the fact that (4.13) and (2.2) imply (2.14) [38], and then using Moser elliptic iterative method [31].

The second step of the proof is to show that

$$h(t,x,y) \leq \frac{C_1}{\mu(B(x,\sqrt{t}))} \exp\left(-\frac{c_1\, d(x,y)^2}{t}\right).$$

This can be done in two ways: 1) using the result of [20], or 2) using the easiest part of Moser parabolic iterative method as in [39, Sect. 5].

The third step is to use the result of the present paper to obtain the lower bound

$$\forall\, t \in (0, \epsilon R^2),\ \ \forall\, y \in B(x, \epsilon\sqrt{t})),\ \ h^B(t,x,y) \geq \frac{c_2}{\mu(B(x,\sqrt{t}))}$$

for some $\epsilon, c_2 > 0$.

The last step is to use the arguments of [16, Sect. 3] (arguments due to Krylov and Safonov [27, 34]) to prove both the parabolic Hölder continuity estimate (2.13) and the parabolic Harnack inequality (2.11).

This proof avoids several technical difficulties related to the parabolic version of Moser iteration.

5 The non-classical case

5.1 Non-classical Gaussian bounds

In this section we consider a connected locally compact non compact complete metric space (M,δ). As above, μ is a positive radon measure on M with full support. We also assume that (M,δ) is a path metric space. That is, any two points $x, y \in M$ there is a continuous map $\gamma : [0, \delta(x,y)] \to M$ such that $\delta(\gamma(s), \gamma(t)) = (t-s)$ for all $0 \leq s < t \leq \delta(x,y)$, see [25]. It follows that each closed metric ball $\{y \in M : \delta(x,y) \leq r\}$ is compact. We say that (M,δ,μ) is doubling if there exists a constant D_1 such that

$$\forall\, x \in M,\ \ \forall\, r > 0,\quad \mu(B_\delta(x,2r)) \leq D_1 \mu(B_\delta(x,r)). \tag{5.1}$$

As we assume that (M,δ) is a length space, (5.1) implies that (2.3), (2.4) and (2.5) are satisfied by δ-balls. In particular, (5.1) implies that there exist positive constants D_1', A such that

$$\forall\, x,y \in M,\ \ \forall\, s > 0,\quad \mu(B_\delta(x,s)) \leq D_1'\, \mu(B_\delta(y,s)) \left(1 + \frac{\delta(x,y)}{s}\right)^A. \tag{5.2}$$

Our aim is to use the technique of Section 3.2 to obtain some results concerning non-classical Gaussian bounds and related Harnack inequalities. Precise definition are given below. These non-classical Gaussian bounds appeared first in the study of Brownian motion on fractals [4]. They have been extended to the case of certain fractal like graphs [5]. We refer to [3] for an excellent introduction to non-classical Gaussian bounds. There is very little literature about such bounds on manifolds but there is no doubt that there are manifolds on which the heat kernel have a non-classical behavior for long time [4]. There are very recent works on random walks on graphs that are closely related in spirit to the material presented here. See [24, 45] and the reference therein.

Assume that $(\mathcal{E}, \mathcal{D}, L^2(M, d\mu))$ is a strictly local regular Dirichlet space associated to a stochastic Hunt process (X_t) on M. Since $(\mathcal{E}, \mathcal{D}, L^2(M, d\mu))$ is strictly local the process (X_t) has continuous paths. One of the tools that will be used below implicitly is the Dynkin-Hunt formula

$$h^B(t,x,y) = h(t,x,y) - E^x\left[h(t-\tau, X_\tau, y)\mathbf{1}_{\{\tau \le t\}}\right] \tag{5.3}$$

for the Dirichlet heat kernel in a fixed ball $B = B_\delta(z,r)$.

For comparison with the setting considered in the rest of this work, note that we have dropped all hypotheses concerning the existence and properties of the intrinsic distance d. This means, in particular, that we will not be allowed to use cut-off function arguments since the distance δ may well have no "gradient" in any reasonable sense. This new setting includes for instance certain Dirichlet spaces on fractals (see e.g., [3]).

To describe non-classical Gaussian bounds, we need some notation. Consider an increasing positive continuous function

$$\rho : (0, +\infty) \to (0, +\infty), \quad \rho(1) = 1. \tag{5.4}$$

The condition $\rho(1) = 1$ is simply a useful normalization. We make two hypotheses on the function ρ.

(R1) There exist $C, c > 0$ and $b_1, b_2 \in (0,1)$ such that

$$\forall\, 0 < t < T,\ \ c(T/t)^{b_1} \le \rho(T)/\rho(t) \le C(T/t)^{b_2}.$$

(R2) For any $t > 0$ the function $s \mapsto s\rho(t/s)$ is an increasing bijection form $(0, +\infty)$ onto itself.

We denote by $s \mapsto G(t,s)$ the inverse function of $u \mapsto u\rho(t/u)$ so that

$$G(t,s)\rho(t/G(t,s)) = s.$$

The purpose of function G is to write down Gaussian like factor of the type $\exp(-cG(t, \delta(x,y))$ in heat kernel estimates. The following consequences of (R1-R2) are noteworthy:

(PR1) $\rho(t) = o(t)$ at infinity.

(PR2) $t \to G(t,s)$ is decreasing and for all $a,t,s > 0$, $G(at, as) = aG(t,s)$.

(PR3) $G(t,s) = 1 \iff \rho(t) = s$ and $G(t,s) \le 1 \iff \rho(t) \ge s$.

(PR4) For each $s > 0$, $G(s,s) = s$. Thus, for all s and $t \in (0,s)$, $G(t,s) \ge s$.

(PR5) There exists a constant D such that, for all $t,s > 0$, $G(t/2, 2s) \le DG(t,s)$

(PR6) Let $C, c > 0$ and $b_1, b_2 \in (0,1)$ be as in (R1). For all $t,s > 0$ with $\rho(t) \le s$, we have

$$\left(\frac{cs}{\rho(t)}\right)^{1/(1-b_1)} \le G(t,s) \le \left(\frac{Cs}{\rho(t)}\right)^{1/(1-b_2)}. \tag{5.5}$$

All these properties are straightforward.

Remark Consider the following weakened version of (R1):

(R1') There exist $C > 0$ and $b_2 \in (0,1)$ such that $\rho(T)/\rho(t) \le C(T/t)^{b_2}$.

The properties (PRi) above holds under the weaker hypothesis (R1')-(R2) except the lower bound in (PR6).

Example 1 If $\rho(t) = t^{1/\alpha}$, $\alpha > 1$, then $G(t,s) = (s^\alpha/t)^{1/(\alpha-1)}$. In particular, for $\alpha = 1/2$, $G(t,s)$ has the classical Gaussian form s^2/t. For example with $\alpha > 2$, see [4, 5].

Example 2 Fix $\alpha_0, \alpha_\infty > 1$ and set $\rho(t) = t^{1/\alpha_0}$ if $t \in (0,1)$, $\rho(t) = t^{1/\alpha_\infty}$ if $t \geq 1$, then

$$G(t,s) = \begin{cases} (s^{\alpha_0}/t)^{1/(\alpha_0-1)} & \text{if } t \in (0,1) \text{ or } t \in (0,s), \\ (s^{\alpha_\infty}/t)^{1/(\alpha_\infty-1)} & \text{if } t \geq 1 \text{ and } t \geq s. \end{cases}$$

On Riemannian manifolds, the value $\alpha_0 = 2$ is forced because the small time asymptotic behavior must be Euclidean. Manifolds with a fractal like skeleton provide examples with $\alpha_\infty > 2$. See again [4, 5].

Define the volume function $(x,t) \mapsto V_{\delta,\rho}(x,t)$ by

$$V_{\delta,\rho}(x,t) = \mu(B_\delta(x,\rho(t))) \tag{5.6}$$

Assuming (PR1')-(PR2), we see that (5.1) implies that $V_{\delta,\rho}$ satisfies

$$\forall\, x \in M,\ t > 0,\ \ V_{\delta,\rho}(x,2t) \leq D_2 V_{\delta,\rho}(x,t). \tag{5.7}$$

We say that $(\mathcal{E}, \mathcal{D}, L^2(M,d\mu))$ satisfies a δ-ρ-Gaussian lower bound if there are constants C_1, c_1 such that,

$$\forall\, x,y \in M,\ t > 0,\ \ h(t,x,y) \geq \frac{c_1}{V_{\delta,\rho}(x,t)} \exp\left(-C_1 G(t,\delta(x,y))\right). \tag{5.8}$$

Similarly, we say an δ-ρ-Gaussian upper bounds holds if,

$$\forall\, x,y \in M,\ t > 0,\ \ h(t,x,y) \leq \frac{C_2}{V_{\delta,\rho}(x,t)} \exp\left(-c_2 G(t,\delta(x,y))\right). \tag{5.9}$$

We say that $(\mathcal{E}, \mathcal{D}, L^2(M,d\mu))$ satisfies a two-sided δ-ρ-Gaussian bound if there exist $C_i, c_i > 0$, $i = 1,2$ such that both (5.8) and (5.9) hold true.

It should be observed that, if we assume that (M,δ,μ) is doubling, then for all t,x,y in the range $1 \leq t \leq \delta = \delta(x,y)$, any (upper, lower, or two-sided) δ-ρ-Gaussian bound is equivalent to

$$\frac{c'}{\mu(B_\delta(x,1))} \exp\left(-C' G(t,\delta)\right). \tag{5.10}$$

In other words, in this range, whether we use $\rho(t)$ or not in the volume factor does not matter. This is because the Gaussian term is smaller than $e^{-c\delta}$ in this range (See (PR4)) and the error produced by replacing $\rho(t)$ by 1 is bounded polynomially in terms of d under (5.1). The expression (5.10) has the advantage to eliminate the meaningless dependence of t in the volume factor.

As a first hint that the above definitions make sense, we offer the following lemma.

Lemma 5.1 *Referring to the above setting and notation, assume that* (5.1) *is satisfied. Assume also that*

$$\forall\, t > 0,\ \forall\, x \in M,\ \forall\, y \in B_\delta(x,\rho(t)),\quad h(t,x,y) \geq \frac{c_0}{\mu(B(x,\rho(t)))}. \tag{5.11}$$

Then the δ-ρ-Gaussian lower bound (5.8) *is satisfied.*

Proof Set $\delta = \delta(x,y)$. If $\delta \leq \rho(t)$, there is nothing to prove. If $\delta \geq \rho(t)$ then $G(t,\delta) \geq 1$. Take n to be the largest integer less or equal to $G(t,3\delta)+1$. By the doubling property of G, $n \approx G(t,\delta)$. Let $\gamma : [0,\delta] \to M$ be a distance minimizing path from x to y and set $x_i = \gamma(i\delta/n)$,

$i = 0, 1, 2, \dots, n$. Then $\delta(x_i, x_{i+1}) = \delta/n$. If we set $B_i = B_\delta(x_i, \rho(t/n))$, by (5.11), there is a constant c_0 such that, for all $i = 1, \dots, n$,

$$\inf_{\xi_i \in B_i, \xi_{i+1} \in B_{i+1}} h(t/n, \xi_i, \xi_{i+1}) \geq c_0 \tag{5.12}$$

since, for $\xi_i \in B_i, \xi_{i+1} \in B_{i+1}$, $\delta(\xi, \xi_{i+1}) \leq 3\delta/n$ and $3\delta/n < \rho(t/n)$. The last inequality follows from (R2), the fact that $3\delta = G(t, 3\delta)\rho(t/G(t, 3\delta))$ and $n \geq G(t, 3\delta)$. By the semigroup property, (5.12) and (5.1), we then have

$$\begin{aligned} h(t,x,y) &\geq \int_{B_1}\int_{B_2}\cdots\int_{B_{n-1}} h(t/n, x, \xi_1)h(t/n, \xi_1, \xi_2)\dots h(t/n, \xi_{n-1}, y)d\xi_1 \dots d\xi_{n-1} \\ &\geq \frac{c_1^n}{V_{\delta,\rho}(x, t/n)}. \end{aligned}$$

By (R1), $V_{\delta,\rho}(x, t/n) \geq n^{-A} V_{\delta,\rho}(x, t)$. Thus, as $n \approx G(t, \delta)$, we obtain

$$h(t,x,y) \geq \frac{c_2^n}{V_{\delta,\rho}(x,t)} \geq \frac{c_3}{V_{\delta,\rho}(x,t)} \exp(-C_3 G(t,\delta))$$

as desired. □

Remark If, in Lemma 5.1, we assume that (5.11) holds only for $0 < t \leq 1$, then we still obtain the lower

$$h(t,x,y) \geq \frac{c_1}{\mu(B(x, \rho(t)))} \exp\left(-C_1 G(t, \delta(x,y))\right)$$

in the range $0 < t \leq 1$ or $1 \leq t \leq \delta(x,y)$. This is because, in the proof, in the range $1 < t \leq \delta(x,y)$, we still have $t/n \leq 1$. Indeed, $n \approx G(t, \delta(x,y))$ and, by , (PR4), $\rho(t/G(t,\delta)) = \delta/G(t,\delta) \leq 1$, hence $t/G(t,\delta) \leq 1$.

Our next observation concerns (5.1).

Lemma 5.2 *Assume that the δ-ρ-Gaussian lower bound* (5.8) *holds. Then there exists D_1 such that* (5.1) *is satisfied.*

Proof Integrate the lower bound for $h(t,x,y)$ over the ball $B_\delta(x, 2r)$ with $r = \rho(t)$. As

$$\int_M h(t,x,y)d\mu(y) \leq 1,$$

it follows that $\mu(B_\delta(x, 2r)) \leq D_1 \mu(B_\delta(x, r))$, for some finite constant D_1. □

Remark Lemma 5.1 and Lemma 5.2 hold true without change if one weakens the hypothesis (R1) to (R1')

5.2 δ-ρ-Parabolic Harnack inequality

We say that $(\mathcal{E}, \mathcal{D}, L^2(M, d\mu))$ satisfies a δ-ρ-parabolic Harnack inequality if there exists a constant C such that, for any real s, any $t > 0$, any $x \in M$, and any solution u of $(\partial_t + L)u = 0$ in $Q^{\delta,\rho} = (s - r, s) \times B_\delta(x, \rho(t))$,

$$\sup_{Q_-^{\delta,\rho}} \{u\} \leq C \inf_{Q_+^{\delta,\rho}} \{u\}$$

where

$$\begin{aligned} Q_+^{\delta,\rho} &= (s - t/4, s) \times B_\rho(x, \rho(t)/2) \\ Q_-^{\delta,\rho} &= (s - 3t/4, s - t/2) \times B_\rho(x, \rho(t)/2). \end{aligned}$$

Theorem 5.3 *Referring to the setting and notation introduced above, the following two properties are equivalent.*

1. *The Dirichlet space $(\mathcal{E}, \mathcal{D}, L^2(M, d\mu))$ satisfies a two-sided δ-ρ-Gaussian bound.*
2. *The Dirichlet space $(\mathcal{E}, \mathcal{D}, L^2(M, d\mu))$ satisfies a δ-ρ-parabolic Harnack inequality.*

Proof of "1 ⇒ 2" To see that a two-sided δ-ρ-Gaussian bound implies an δ-ρ-parabolic Harnack inequality, the argument of [16, Sect 3] can be adapted. See also [36]. Note that the mentioned argument is based on Formula (5.3). The details are omitted but we would like to point out that the full hypothesis (R1) is used here. It seems that (R1') is not sufficient to run the relevant argument of [16, Sect 3].

We now turn to the proof of "2 ⇒ 1". This implication holds true under the weaker hypothesis (R1'). We start with the following Lemma.

Lemma 5.4 *If $(\mathcal{E}, \mathcal{D}, L^2(M, d\mu))$ satisfies a δ-ρ-parabolic Harnack inequality then there exist positive constants c, C such that*

$$cV_{\delta,\rho}(x,t)^{-1} \leq h(t,x,x) \leq CV_{\delta,\rho}(x,t)^{-1}$$

with $V_{\delta,\rho}$ defined at (5.6). *In particular, the doubling inequality* (5.1) *is satisfied.*

Proof Use the argument given in [38, pg 32] for the case when $\rho(t) = t^{1/2}$. The adaptation to general ρ is straightforward. □

Proposition 5.5 *If $(\mathcal{E}, \mathcal{D}, L^2(M, d\mu), \delta)$ satisfies a δ-ρ-parabolic Harnack inequality then it satisfies the δ-ρ-Gaussian lower bound* (5.8).

Proof Use Lemma 5.1 and Lemma 5.4. □

Proposition 5.6 *If $(\mathcal{E}, \mathcal{D}, L^2(M, d\mu))$ satisfies a δ-ρ-parabolic Harnack inequality then it satisfies the δ-ρ-Gaussian upper bound* (5.9).

We start the proof with the following Lemma.

Lemma 5.7 *Assuming the δ-ρ-parabolic Harnack inequality holds, there exists $c_0 > 0$ such that for all $t > 0$ and all $x \in M$, we have*

$$\int h^B(t,x,z)d\mu(z) \geq c_0$$

where $B = B_\delta(x, \rho(t))$.

Proof The desired inequality easily follows from the postulated δ-ρ-parabolic Harnack inequality. To see this, consider the solution u of $(\partial_t + L)u = 0$ obtained by setting $u(s,y) = \int h^B(s,y,z)d\mu(z)$ for $s > 0$, $y \in B$ and $u(s,y) = 1$ if $s \leq 0$, $y \in B$. □

Lemma 5.8 *Assume that there exists $c_0 > 0$ such that for all $t > 0$ and all $x \in M$, we have $\int h^B(t,x,z)d\mu(z) \geq c_0$ where $B = B_\delta(x, \rho(t))$. Then there exists $C_1, c_1 > 0$ such that, for all $x \in M$ and all $t > 0$,*

$$\int e^{c_1\delta(x,z)/\rho(t)} h(t,x,z)d\mu(z) \leq C_1.$$

Proof Consider the Hunt process $(X_s)_{s\geq 0}$ associated to the Dirichlet space $(\mathcal{E}, \mathcal{D}, L^2(M, d\mu))$. For $k = 1, 2, \ldots, n$, let T_k be the stopping time

$$T_k = \inf\{s > 0 : \delta(x, X_s) \geq k\rho(t)\}.$$

Set also

$$T_z = \inf\{s > 0 : \delta(z, X_s) \geq \rho(t)\}$$

and note that

$$\sup_z \sup_{s \leq t} \mathbf{P}_z(T_z \leq s) \leq 1 - c_0.$$

By the strong Markov property (denoting by $(\Xi, \mathbf{P}_y)$ the underlying Probability space), we have

$$\begin{aligned}
\mathbf{P}_x(T_k \leq t) &= E_x(\mathbf{P}_x(T_k \leq t/\mathcal{F}_{T_{k-1}})) \\
&= \int_\Xi \mathbf{1}_{\{T_{k-1} \leq t\}}(\xi)\, \mathbf{P}_{X_{T_{k-1}}(\xi)}(T_k \leq t - T_{k-1}(\xi))\, d\mathbf{P}_x(\xi) \\
&\leq \mathbf{P}_x(T_{k-1} \leq t) \sup\{\mathbf{P}_z(T_z \leq s) : z \in M,\ s \leq t\} \\
&\leq (1 - c_0)\mathbf{P}_x(T_{k-1} \leq t).
\end{aligned}$$

By induction, we get

$$\mathbf{P}_x(T_n \leq t) \leq (1 - c_0)^n.$$

Now, the desired conclusion follows from

$$\int_{\{\delta(x,z) \geq n\rho(t)\}} h(t, x, z) d\mu(z) \leq \mathbf{P}_x(T_n \leq t)$$

and

$$\begin{aligned}
\int e^{c\delta(x,z)/\rho(t)} h(t, x, z) d\mu(z) &\leq \sum_{n \geq 0} e^{c(n+1)} \int_{\{\delta(x,z) \geq n\rho(t)\}} h(t, x, z) d\mu(z) \\
&\leq \sum_{n \geq 0} e^{c(n+1)} (1 - c_0)^n
\end{aligned}$$

since the last series converges for c small enough. □

Lemma 5.9 *Assume that there exists $c_0 > 0$ such that for all $t > 0$ and all $x \in M$, $t > 0$, we have $\int h^B(t, x, z) d\mu(z) \geq c_0$ when $B = B_\delta(x, \rho(t))$. Fix $N > 0$. Then there exists $C_2, c_2 > 0$ such that, for any $R > 0$, any $x \in M$ and any $t > 0$,*

$$\int_{\{\delta(x,y) \geq R\}} \left(1 + \frac{\delta(x, y)}{\rho(t)}\right)^N h(t, x, y) d\mu(y) \leq C_2 \exp\left(-c_2 G(t, R)\right).$$

Proof By Lemma 5.8, the semigroup property and the triangle inequality, for all $n = 1, 2, \ldots$,

$$\int e^{c_1\delta(x,z)/\rho(t)} h(nt, x, z) d\mu(z) \leq C_1^n.$$

Changing t to t/n, we obtain

$$\begin{aligned}
\int_{\{\delta(x,z) \geq R\}} \left(1 + \frac{\delta(x, z)}{\rho(t)}\right)^N h(t, x, z) d\mu(z) &\leq \int_{\{\delta(x,z) \geq R\}} \left(1 + \frac{\delta(x, z)}{\rho(t/n)}\right)^N h(t, x, z) d\mu(z) \\
&\leq C_1^n \exp\left(-c_2 R/\rho(t/n)\right) \\
&= \exp\left(C_2[n\rho(t/n) - c_3 R]/\rho(t/n)\right)
\end{aligned}$$

with $C_2 = \log C_1$ and $c_3 = c_2/C_2$. If $c_3R/2 \leq \rho(t)$, $G(t, R)$ is bounded above and taking $n = 1$ gives the desired inequality. If $c_3R/2 \geq \rho(t)$, then $G(t, c_3R/2) \geq 1$. Take n to be the smallest integer such that $n \leq G(t, c_3R/2)$. Then $n\rho(t/n) - c_3R \leq -c_3R/2$ and

$$\frac{[c_3R/2]}{\rho(t/n)} \geq \frac{c[c_3R/2]}{\rho(t/(n+1))} \geq \frac{c[c_3R/2]}{\rho(t/G(t, c_3R/3))} = cG(t, c_3R/2).$$

Hence, we obtain

$$\int_{\{\delta(x,z)\geq R\}} \left(1+\frac{\delta(x,z)}{\rho_\omega(t)}\right)^N h(t,x,z)d\mu(z) \leq \exp\left(-cC_2G(t,c_3R/2)\right) \leq \exp\left(-C_4G(t,R)\right)$$

where the last inequality follows from the doubling property of G. □

Proof of Proposition 5.6 Fix $x,y \in M$ and $t>0$. Let $R=\delta(x,y)/2$. By Lemma 5.4, the inequality $h(t,\xi,\zeta) \leq \sqrt{h(t,\xi,\xi)h(t,\zeta,\zeta)}$ and (5.2), there exists C, A such that

$$h(s,\xi,\zeta) \leq \frac{C}{V_{\delta,\rho}(\xi,s)}\left(1+\frac{\delta(\xi,\zeta)}{\rho(s)}\right)^A. \tag{5.13}$$

Now, write

$$\begin{aligned} h(2t,x,y) &= \int h(t,x,z)h(t,z,y)d\mu(z) = \int_{B_\delta(x,R)\cup B_\delta(x,r)^c} h(t,x,z)h(t,z,y)d\mu(z) \\ &\leq \int_{B_\delta(y,R)^c} h(t,x,z)h(t,z,y)d\mu(z) + \int_{B_\delta(x,R)^c} h(t,x,z)h(t,z,y)d\mu(z). \end{aligned}$$

By (5.13) and the triangle inequality, we have

$$\begin{aligned} &\int_{B_\delta(x,R)^c} h(t,x,z)h(t,z,y)d\mu(z) \\ &\qquad \leq \frac{C}{V_{\delta,\rho}(x,t)}\left(1+\frac{\delta(x,y)}{\rho(t)}\right)^A \int_{B_\delta(x,R)^c}\left(1+\frac{\delta(y,z)}{\rho(t)}\right)^A h(t,y,z)d\mu(z) \\ &\qquad \leq \frac{C'}{V_{\delta,\rho}(x,t)}\left(1+\frac{\delta(x,y)}{\rho(t)}\right)^A \exp\left(-c_2G(t,R)\right) \\ &\qquad \leq \frac{C'}{V_{\delta,\rho}(x,t)} \exp\left(-c_2'G(t,\delta(x,y))\right) \end{aligned}$$

By symmetry and (5.2), we get the same bound for $\int_{B(y,R)^c} h(t,x,z)h(t,z,y)d\mu(z)$. Thus, for all $t>0$ and $x,y\in M$, we have proved the bound

$$h(t,x,y) \leq \frac{C}{V_{\delta,\rho}(x,t)} \exp\left(-cG(t,\delta(x,y))\right).$$

This finishes the proof of Proposition 5.6. □

Together, Lemma 5.4 and Proposition 5.6 prove Theorem 5.3.

5.3 δ-Elliptic and δ-ρ-parabolic Harnack inequalities

We say that $(\mathcal{E},\mathcal{D},L^2(M,d\mu))$ satisfies a δ-elliptic Harnack inequality if there exists a constant C such that any non-negative solution u of $\Delta u = 0$ in a ball $B = B_\delta(x,r)$ satisfies

$$\sup_{B_\delta(x,r/2)}\{u\} \leq C \inf_{B_\delta(x,r/2)}\{u\}. \tag{5.14}$$

We make the following straightforward but important observations:

- The δ-ρ-parabolic Harnack inequality for $(\mathcal{E},\mathcal{D},L^2(M,d\mu),\delta)$ implies the elliptic Harnack inequality on δ-balls.

- The elliptic Harnack inequality for δ-balls implies the corresponding δ-Hölder continuity: There exist two positive real A, α such that for any ball $B = B_\delta(x, r)$ and any harmonic function u in B,

$$\forall\, x, y \in \frac{1}{2}B,\ \ |u(x) - u(y)| \le A(\delta(x, y)/r)^\alpha \sup_B\{|u|\}. \tag{5.15}$$

Since we will not use it below, we leave to the reader the simple task to formulate the adequate Hölder continuity statement that follows from a δ-ρ-parabolic Harnack inequality.

In this section, we establish the following theorem.

Theorem 5.10 *Assume that* $(\mathcal{E}, \mathcal{D}, L^2(M, d\mu))$ *satisfies the* δ*-elliptic Harnack inequality* (5.14) *and that*

$$\int_M h(t, x, y)d\mu(y) = 1,$$

for some (equivalently, any) $t > 0$*. Assume further that* $(\mathcal{E}, \mathcal{D}, L^2(M, d\mu))$ *satisfies the doubling condition* (5.1) *and the* δ-ρ*-Gaussian upper bound* (5.9)*, that is,*

$$\forall\, x, y \in M,\ \ \forall\, t > 0,\ \ h(t, x, y) \le \frac{C_2}{V_{\delta,\rho}(x, t)} \exp\left(-c_2\, G(t, \delta(x, y))\right).$$

Then the δ-ρ*-parabolic Harnack inequality holds and so does the two-sided* δ-ρ*-Gaussian bound.*

The proof is essentially the same as in Section 4.2. The heart of the matter is the following result.

Proposition 5.11 *Under the assumption of* Theorem 5.10*, there exist* $\epsilon_i > 0$*,* $i = 1, 2, 3$*, such that the heat kernel* $h(t, x, y)$ *satisfies the lower bound*

$$\forall\, x \in M,\ \ \forall\, t > 0,\qquad \inf_{y \in B_\delta(x, \epsilon_1 \rho(t))}\{h(t, x, y)\} \ge \frac{\epsilon_2}{\mu(B_\delta(x, \rho(t)))}.$$

Proof We follow the line of reasoning of the proof of Proposition 3.5. First, we need a lower bound for the Dirichlet eigenvalue given in the following lemma.

Lemma 5.12 *Assume that* (5.1) *and* (5.9) *are satisfied. Then, there is* $c > 0$ *such that*

$$\forall\, x \in M,\ t > 0,\ \ \lambda_1(B_\delta(x, \rho(t))) \ge ct^{-1}.$$

Proof In fact, we only need to assume (5.1) and the on-diagonal upper bound

$$h(t, x, x) \le \frac{C_1}{V_{\delta,\rho}(x, t)}. \tag{5.16}$$

Indeed, fix $x \in M$, $t > 0$, and set $B = B_\delta(x, \rho(t))$. Let $T \ge t$ be such that $\rho(T) = k\rho(t)$ for some k to be chosen later. By (5.16) and (5.1), we find that

$$h^B(T, z, z) \le \max_{z \in B}\{h(T, z, z)\} \le \frac{C}{\mu(B_\delta(x, k\rho(t)))}.$$

Thus, by (2.16), there exists $\gamma > 0$ such that

$$\int_B h(T, z, z)d\mu(z) \le \frac{C\mu(B_\delta(x, \rho(t))}{\mu(B_\delta(x, k\rho(t))} \le \frac{C'}{k^\gamma}.$$

Moreover, expending $h(t,z,z)$ along a Dirichlet eigenfunction orthonormal basis of $L^2(B)$, we get

$$e^{-T\lambda_1(B)} \le \sum_1^\infty e^{-T\lambda_i} = \int_B h^B(T,z,z)d\mu(z) \le \frac{C'}{k^\gamma}.$$

If k is chosen so large that $C'/k^\gamma = e^{-1}$, we get

$$T\lambda_1(B) \ge 1.$$

By (R1), once k is fixed, there is a constant C such that $T \le Ct$. □

Next, we state the analog of Lemma 3.9. The proof is the same, up to changes of notation.

Lemma 5.13 *Assume the hypothesis of* Theorem 5.10 *are satisfied. Let* $B = B_\delta(x, \rho(t))$ *and* $V = V_{\delta,\rho}(x,t)$, $t > 0$. *Then the following estimates hold:*

1. *There exists* C_1 *such that for all* $z, y \in B$,

$$h^B(s,y,z) \le \frac{C_2}{V_{\delta,\rho}(y,s)} \exp\left(-c_2 G(s, \delta(z,y))\right). \tag{5.17}$$

2. *There exists* C_2 *such that*

$$\forall\, z, y \in B, \quad |\partial_t h^B(t,z,y)| \le \frac{C_A}{tV}. \tag{5.18}$$

3. *There exists* $\epsilon, a_1 > 0$, *and, for any* $0 < \theta < 1$, *there exists* $C_{A,\theta}$ *such that*

$$\forall\, s \in (\theta t, +\infty)\, \forall\, z, y \in B, \quad h^B(s,y,z) \le \frac{C_{A,\theta}}{V} \exp\left(-\frac{a_1 s}{A^2 t}\right). \tag{5.19}$$

Lemma 5.13 is used to prove Lemma 5.14 below from which, in turns, Proposition 5.11 easily follows.

Lemma 5.14 *Under the assumption of* Theorem 5.10, *for any* $\sigma > 0$ *and any* $A \ge 1$ *there exist two positive reals* $C_{\sigma,A}$ *and* ϵ_A *such that for all* $x \in M$, $t \in (0, \epsilon_A R^2)$, *and* $y \in B$,

$$|h^B(t,x,y) - h^B(t,x,x)| \le \left[\sigma + C_{\sigma,A}\left(\frac{\delta(x,y)}{\rho(t)}\right)^\alpha\right]\frac{1}{V}$$

where $B = B(x, A\rho(t))$, $V = \mu(B_\delta(x,\rho(t)))$ *as above, and* α *is the Hölder exponent in* (5.15).

The proof of 5.14 is the same as the proof of Lemma 3.8 in Section 3.4. It is omitted. Of course, $\sqrt{t}$ must be changed to $\rho(t)$ everywhere and (PR6) is used to estimate integrals.

We end this section with the following lemma which may be of some independent interest.

Lemma 5.15 *Assume that either a two-sided* δ-ρ-*Gaussian bound or an* δ-ρ-*parabolic Harnack inequality holds. Then for any ball* $B = B_\delta(x,\rho(t)) \subset M$, *the lowest Dirichlet eigenvalue* $\lambda_1(B)$ *is bounded above and below by*

$$at^{-1} \le \lambda_1(B) \le At^{-1}.$$

Similarly, the lowest non-zero Neumann eigenvalue $\lambda_1^N(B)$ *of any ball* B *as above satisfies*

$$at^{-1} \le \lambda_1^N(B) \le At^{-1}.$$

Proof For the Dirichlet eigenvalue, the lower bound comes from the proof of Lemma 5.12. Indeed, we proved that (5.16) suffices to imply the desired lower bound. Obviously, (5.16) follows from an δ-ρ-Gaussian upper bound. It also easily follow from an δ-ρ-parabolic Harnack inequality.

For the Neumann eigenvalue, the method of [28, 38] can be used, together with a covering argument due to [26], to prove the lower bound. In the present setting, the original argument of [26] should be somewhat modified (see e.g., [17, Th 5.4] and [40, Sec 5.3]).

The upper bounds are more interesting as the obvious test function argument fails to yield the desired result. One can proceed as follows. Reproducing the proof of (3.6), one sees that

$$\forall\, z, y \in B_\delta(x, \rho(t)), \quad h^B(2t, y, z) \le \frac{C_1}{\mu(B)} \exp\left(-\lambda_1(B)t\right).$$

Thus,

$$h^B(2t, x, x) \le \frac{C_1}{\mu(B)} \exp\left(-\lambda_1(B)t\right).$$

But our hypothesis also implies that

$$h^B(2t, x, x) \ge \frac{c_1}{\mu(B)}.$$

Thus, $t\lambda_1(B) \le \log(C_1/c_1)$ as desired.

For the Neumann eigenvalue, one can used the principal Dirichlet eigenfunctions of two disjoint balls contained in B and of radius of order $\rho(t)$ to obtain a function f supported in B such that $\int_B f d\mu = 0$ and $\int_B |f|^2 d\mu \ge ct \int_B |\nabla f|^2 d\mu$. This shows that $\lambda_1^N(B) \le (ct)^{-1}$. □

References

[1] Auscher P. and Coulhon T. *Gaussian bounds for random walks from elliptic regularity.* Ann. Inst. Henri Poincaré, Prob. Stat., 35, 1999, 605-630.

[2] Bakry D., Coulhon T., Ledoux M. and Saloff-Coste L. *Sobolev Inequalities in Disguise.* Indiana Univ. Math. J., 44, 1995, 1033-1073.

[3] Barlow M. *Diffusions on fractals.* in "Lectures in Probability Theory and StatisticsEcole d'été de Probabilitś de Saint Flour XXV–1995", Lecture Notes in Math. 1690, Springer, 1998, 1-121.

[4] Barlow M. and Bass R. *Transition densities for Brownian motion on the Sierpinski carpet.* Probab. Th. Rel. Fields, 91 (1992) 307-330.

[5] Barlow M. and Bass R. *Random walks on graphical Sierpinski carpets.*

[6] Bendikov A. and Saloff-Coste L. *On and off-diagonal heat kernel behaviors on certain infinite dimensional local Dirichlet spaces.* Preprint, 1999.

[7] Carron G. *Inégalités isopérimétriques de Faber-Krahn et conséquences.* In Actes de la table ronde de géométrie différentielle en l'honneur de Marcel Berger, Soc. Math. France, Séminaires et Congrés 1, 1996, 205-232.

[8] Cheeger J., Gromov M. and Taylor M. *Finite propagation speed, kernel estimates for functions of the Laplace operator, and the geometry of complete Riemannian manifolds.* J. Diff. Geom. 17, 1982, 15-53.

[9] Coulhon T. and Grigor'yan A. *On-diagonal lower bounds for heat kernels and Markov chains.* Duke Math. J. 89, 1997, 133-199.

[10] Coulhon T. and Saloff-Coste L. *Variétés Riemanniennes isométriques à l'infini.* Rev. Mat. Iberoamericana, 11, 1995, 687-726.

[11] Davies E.B. *Heat kernels and spectral theory.* Cambridge University Press, 1989.

[12] Davies E.B. *Heat kernel bounds, conservation of probability and the Feller property.* J. d'Analyse Math., 58, 1992, 99-119.

[13] Davies E.B. *Non-Gaussian aspects of Heat kernel behaviour.* J.London Math. Soc. 55, 1997, 105-125.

[14] Delmotte T. *Parabolic Harnack inequality and estimates of Markov chains on graphs.* Rev. Mat. Iberoamericana, 15, 1999, 181-232.

[15] Delmotte T. *Elliptic and parabolic Harnack inequalities.* Potential Analysis, , to appear.

[16] Fabes E. and Stroock D. *A new proof of Moser's parabolic Harnack inequality using the old ideas of Nash.* Arch. Rat, Mech. Anal. 96, 1986, 327-338.

[17] Franchi B., Gutiérrez C. and Wheeden R. *Weighted Sobolev-Poincaré inequalities for Grushin type operators* Comm. in Partial Differential Equations, 19, 1994, 523-604.

[18] Fukushima M., Oshima Y. and Takeda, M. *Dirichlet forms and Symmetric Markov processes,* W. De Gruyter, 1994.

[19] Grigor'yan A. *The heat equation on non-compact Riemannian manifolds.* Matem. Sbornik, 182, 1991, 55-87. Engl. Transl. Math. USSR Sb., 72, 1992, 47-77.

[20] Grigor'yan A. *Heat kernel upper bounds on a complete non-compact Riemannian manifold.* Revista Mat. Iberoamericana, 10, 1994, 395-452.

[21] Grigor'yan A. *Gaussian upper bounds for the heat kernel on arbitrary manifolds.* J. Differential Geometry, 45, 1997, 33-52.

[22] Grigor'yan A. *Analytic and geometric background of recurrence and non-explosion of the Brownian motion on Riemannian manifolds.* Bull. A.M.S., 36, 1999, 135-249.

[23] Grigor'yan A. and Saloff-Coste L. *Heat kernel on connected sums of Riemannian manifolds.* Mathematical Research Letters 6, 1999, 1-14.

[24] Grigor'yan A and Telcs A. *Sub-Gaussian estimates of heat kernels on infinite graphs.* Preprint 2000.

[25] Gromov M. *Metric structures for Riemannian and non-Riemannian spaces.* Birkhäuser, 1998.

[26] Jerison D. *The Poincaré inequality for vector fields satisfying the Hörmander's condition.* Duke Math. J. 53, 1986, 503-523.

[27] Krylov N. and Safonov M. *A certain property of solutions of parabolic equations with measurable coefficients.* Math. USSR-Izs. 16, 1981, 151-164.

[28] Kusuoka S. and Stroock D. *Applications of Malliavin Calculus, Part 3.* J. Fac. Sci. Univ. Tokyo, Sect. IA Math. 34, 1987, 391-442.

[29] Kuzmenko Y.T. and Molchanov S.A. *Counterexamples to Liouville-type theorems* Vestnik Moskov. Univ. Ser. I Mat. Mekh., 6, (1976) 39-43. Engl. Transl. Moscow Univ. Math. Bull., 34, 1979, 35-39.

[30] Li, P. and Yau S-T., *On the parabolic kernel of Schrödinger operator.* Acta Math. 156, 1986, 153-201.

[31] Moser J. *On Harnack's theorem for elliptic differential equations.* Comm. Pure Appl. Math., 14, 1961, 577-591.

[32] Moser J. *A Harnack inequality for parabolic differential equations.* Comm. Pure Appl. Math., 16, 1964, 101-134. Correction in, 20, 1967, 231-236.

[33] Moser J. *On a pointwise estimate for parabolic differential equations.* Comm. Pure Appl. Math., 24, 1971, 727-740.

[34] Safonov M. *Harnack's inequality for elliptic equations and the Hölder property of their solutions*, J. Soviet Math. 21, 1983, 851-863.

[35] Saloff-Coste L. *Analyse sur les groupes à croissance pôlynomiale* Ark. för Mat. 28, 1990, 315-331.

[36] Saloff-Coste L. and Stroock D. *Opéarteurs uniformément sous-elliptiques sur les groupes de Lie.* J. Funct. Anal. 98, 1991, 97-121.

[37] Saloff-Coste L. *Uniformly elliptic operators on Riemannian manifolds.* J. Diff. Geom. 36, 1992, 417-450.

[38] Saloff-Coste L. *A note on Poincaré, Sobolev and Harnack inequalities.* Duke Math. J., IMRN, 2, 1992, 27-38.

[39] Saloff-Coste L. *Parabolic Harnack inequality for divergence form second order differential operators.* Potential Analysis, 4, 1995, 429-467.

[40] Saloff-Coste L. *Some aspects of Sobolev inequalities.* Lecture Notes of a Graduate Course, Cornell University, 1999.

[41] Sturm K-T. *On the geometry defined by Dirichlet forms.* In Seminar on Stochastic Processes, Random Fields and Applications, Ascona, E. Bolthausen et al. Ed. Birkhäuser–Progress in Probability, Vol 36, 1995, 231-242.

[42] Sturm K-T. *Analysis on local Dirichlet spaces I: Recurrence, conservativeness and L^p-Liouville properties.* J. Reine Angew. Math. 456, 1994, 173-196.

[43] Sturm K-T. *Analysis on local Dirichlet spaces II. Upper Gaussian estimates for fundamental solutions of parabolic equations.* Osaka J. Math. 32, 1995, 275-312.

[44] Sturm K-T. *Analysis on local Dirichlet spaces III. The parabolic Harnack inequality.* J. Math. Pures Appl. 75, 1996, 273-297.

[45] Telcs A. *Local sub-Gaussian estimates of heat kernels on graphs, the strongly recurrent cases.* Preprint 2000.

[46] Varopoulos N. *Fonctions harmoniques sur les groupes de Lie.* CR. Acad. Sci. Paris, Sér. I Math. 304, 1987, 519-521.

[47] Varopoulos N. *Small time Gaussian estimates of the heat diffusion kernel, Part 1: the semigroup technique.* Bull. Sci. math. 113, 1989, 253-277.

[48] Varopoulos, Saloff-Coste and Coulhon, *Analysis and geometry on groups.* Cambridge University Press, 1993.