Spectral multipliers on metabelian groups

by Waldemar Hebisch[1]

Institute of Mathematics, Wrocław University,

pl. Grunwaldzki 2/4, 50-384 Wrocław, Poland.

e-mail: `hebisch@math.uni.wroc.pl`

www: `http://www.math.uni.wroc.pl/~hebisch/`

Introduction

Let G be a Lie group, X_j right invariant vector fields on G, which generate (as a Lie algebra) the Lie algebra of G,

$$L = -\sum X_j^2.$$

Then L is called sublaplacian, and it well-known that L is positive definite and essentially selfadjoint on $C_c^\infty(G) \subset L^2(G)$, where $L^2(G)$ is taken with respect to a left-invariant Haar measure dg. By the spectral theorem, for any bounded Borel measurable function $F : [0, \infty) \mapsto \mathbb{C}$ the operator $F(L)f = \int_0^\infty F(\lambda)dE(\lambda)f$ is bounded on $L^2(G)$. We are interested in the behavior of $F(L)$ on L^p.

This question has a long history. Classical results for polynomial growth case are [15], [16], [5], [18], [1], [7], [22] for exponential growth [8], [23], [2], [3]. Newer results show that connection with growth is more complicated [11], [19], [12], [10], [9], [4], [6], [13], [20], [14], [21], [17].

In this paper we consider $L^1(G)$ boundedness of $F(L)$ for (some) metabelian G and a distinguished L on G. Of the main interest is that the group is of exponential growth, and possibly higher rank. Previously positive

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results about higher rank groups where only about Iwasawa type groups. Also, our groups may be unimodular, so it is the second positive result (after [13]) about unimodular groups, and the first giving a family of examples.

Results

Let $G = \mathbb{R}^n \ltimes \mathbb{R}^m$, adjoint action is semisimple, $L = L_0 + L_1$, L_0 lives on $\mathbb{R}^n$, L_1 lives on $\mathbb{R}^m$ and is a sum of (squares of) eigenvectors for adjoint action. More precisely, assume that λ_j, $j = 1, \ldots, m$ are linear forms on $\mathbb{R}^n$, e_j, $j = 1, \ldots, m$ is the canonical basis of $\mathbb{R}^m$, linear operator $A(x) : \mathbb{R}^m \mapsto \mathbb{R}^m$ is given by the formula $A(x)e_j = \lambda_j(x)e_j$ and

$$(x_1, y_1)(x_2, y_2) = (x_1 + x_2, \exp(A(-x_2))y_1 + y_2).$$

The right-invariant vector fields are:

$$X_j = \partial_{x_j}$$

and

$$Y_j = \exp(-\lambda_j(x))\partial_{y_j}.$$

We assume that

$$L = -\sum X_j^2 - \sum Y_j^2 = L_0 + L_1.$$

We can transform general L_0 to our form, but for L_1 the assumption is somewhat restrictive.

In this paper we identify convolution operators with functions:

$$\exp(-tL)f = \exp(-tL) * f.$$

(1.1). **Theorem**. *If G and L are as above, then there exists C such that*

$$\|\exp(-(1 + is)L)\|_{L^1} \leq C(1 + |s|^{3m+n}).$$

(1.2). **Theorem**. *For every compactly supported* $F \in C^{3m+n+1}$ *the operator* $F(L)$ *is bounded on* $L^1(G)$.

Theorem (1.2) is a straightforward consequence of (1.1) .

Before the proof of (1.1) we need a lemma about "symbols". We consider it as well-known, but the form given below is adjusted to our needs.

(1.3). **Lemma**. *There is* C *such that if* E *is a normed space,* $f : \mathbb{R} \mapsto E$, $|f|$ *is integrable,* $b \geq 1$,

$$\sup |\hat{f}| \leq a,$$

$$\sup |\omega \partial_\omega \hat{f}(\omega)| \leq ab,$$

$$\sup |\omega^2 \partial_\omega^2 \hat{f}(\omega)| \leq ab^2,$$

then

$$|f(x)| \leq \frac{Cab}{|x|}.$$

Remark The lemma remains valid as long as $\hat{f}$ is reasonably defined (like $f \in S(\mathbb{R}, E^*)^*$, where $S(\mathbb{R}, E^*)$ consists of E^* valued Schwartz class functions).

Proof: Let $\phi \in C^\infty(\mathbb{R})$ be such that $\phi(x) = 1$ for $|x| \leq 1$ and $\phi(x) = 0$ for $|x| \geq 2$. Fix $x_0 \neq 0$ and let $r = \frac{b}{|x_0|}$. Put $\hat{f}_1(\omega) = \phi(\omega/r)\hat{f}(\omega)$ and $\hat{f}_2(\omega) = (1 - \phi(\omega/r))\hat{f}(\omega)$. We have

$$|f_1(x)| \leq \int |\hat{f}_1| d\omega \leq \int_{-2r}^{2r} a d\omega = 4ar,$$

and

$$|x^2 f_2(x)| \leq \int |\partial_\omega^2 \hat{f}_2(\omega)| d\omega.$$

By the Leibnitz formula

$$\partial_\omega^2 \hat{f}_2(\omega) = (1-\phi(\omega/r))\partial_\omega^2 \hat{f}(\omega) - 2r^{-1}\phi'(\omega/r)\partial_\omega \hat{f}(\omega) + r^{-2}\phi''(\omega/r)\hat{f}(\omega)$$

so

$$\int |\partial_\omega^2 \hat{f}_2(\omega)| d\omega \le \int\limits_{|\omega|>r} \frac{ab^2}{\omega^2} d\omega + \int\limits_{2r>|\omega|>r} 2Cr^{-1}ab\omega^{-1} d\omega + \int\limits_{2r>|\omega|>r} Cr^{-2}a d\omega$$

$$\le ab^2 r^{-1} + 4Cabr^{-1} + 2Car^{-1} \le C'ab^2 r^{-1}.$$

Now

$$|f(x_0) \le |f_1(x_0)| + |f_2(x_0)| \le 4ar + C'ab^2r^{-1}|x_0|^{-2}$$

$$= (4+C')ab|x_0|^{-1} = C''ab|x_0|^{-1}.$$

△

Proof: of (1.1) . We decompose the regular representation of G using Fourier transform in y variable. In coordinates

$$L = -\Delta_x - \sum \exp(-2\lambda_j(x))\partial_{y_j}^2$$

where $\Delta_x = \sum \partial_{x_j}^2$.

If we denote by H_z the Fourier transform (in y variable) of L at z, then

$$H_z = -\Delta_x + \sum z_j^2 \exp(-2\lambda_j(x)).$$

$\Re H_z \ge 0$, provided that $\Re z_j > \Im z_j$, $j = 1, \dots, m$, so $z \mapsto \exp(-tH_z)$ is bounded holomorphic in the area given by the inequalities.

Considering $(t+is)H_z$ we see that $\exp(-(t+is)H_z)$ is bounded and holomorphic as long as $\Re(t+is)z_j^2 \ge 0$, $j = 1, \dots, m$. Moreover, we can estimate the integral kernels

$$\| \exp(-(2t+is)H_z)\delta_0\|_{L^2} \le \| \exp(-(t+is)H_z)\| \| \exp(-tH_z)\delta_0\|_{L^2}.$$

By the Feynmann-Kac formula

$$\|\exp(-tH_z)\delta_0\|_{L^2} \le \|q_t\|_{L^2} = ct^{-n/4}$$

where q_t is ordinary euclidean heat kernel.

Consequently, by the Cauchy integral formula (for real z)

$$\|\partial_z^\alpha \exp(-(t+is)H_z)\delta_0\|_{L^2} \le C_\alpha |z_1|^{-\alpha_1} \cdot \ldots \cdot |z_m|^{-\alpha_m} (1+\frac{|s|}{t})^{|\alpha|} t^{-n/4}$$

Applying (1.3) m times we get

$$\|\exp(-(t+is)L)(\cdot,y)\|_{L^2} \le C''(|y_1| \cdot \ldots \cdot |y_m|)^{-1}(1+\frac{|s|}{t})^m t^{-n/4}.$$

In [14] (as the first step in proof of Theorem (1.1)) we proved that

$$\int |\exp(-(1+is)L)(g)| e^{d(g,0)} dg \le C\exp(Cs^2) \tag{1.4}$$

where $d(x,y)$ is the optimal control distance associated to L. One easily checks that

$$\{g : d(g,0) < r\} \subset \{(x,y) : |x| < r, |y| < c_d \exp(c_d r)\}.$$

To estimate L^1 norm we put $r = Cs^2$, $c = c_d C$, $A_j = \{(x,y) : |x| < Cs^2, |y_j| < \exp(-mcs^2), |y_l| < \exp(cs^2), l \ne j\}$. Note $|A_j| \le Cs^{2n}$. We have

$$\|\exp(-(1+is)L)\|_{L^1} \le \int_{d(g,0)>r} |\exp(-(1+is)L)(g)| dg$$

$$+ \int_{|x|<cs^2, \exp(-mcs^2) \le |y_j| \le \exp(cs^2)} |\exp(-(1+is)L)((x,y))| dxdy$$

$$+ \sum_j \int_{A_j} |\exp(-(1+is)L)|(g) dg$$

$$= I_\infty + I_0 + \sum I_j.$$

For I_∞ we use exponential estimate (1.4)

$$\int_{d(g,0)>r} |\exp(-(1+is)L)(g)|dg \le e^{-r} \int |\exp(-(1+is)L)(g)| \exp(d(g,0))dg$$

$$\le \exp(-Cs^2) C \exp(Cs^2) = C.$$

Next

$$I_j \le |A_j|^{1/2} \| \exp(-(1+is)L)\|_{L^2} \le C|s|^n.$$

Finally

$$I_0 = \int_{\exp(-mcs^2)\le|y_j|\le\exp(cs^2)} \int_{|x|<Cs^2} |\exp(-(1+is)L)(x,y)|dxdy$$

$$\le \int_{\exp(-mcs^2)\le|y_j|\le\exp(cs^2)} |\{x : |x| < cs^2\}|^{1/2} \| \exp(-(1+is)L)(\cdot,y)\|_{L^2} dy$$

$$\le \int_{\exp(-mcs^2)\le|y_j|\le\exp(cs^2)} cs^n C''(|y_1| \cdot \ldots \cdot |y_m|)^{-1}(1+|s|)^m dy$$

$$\le C|s|^n(1+|s|)^m \left(2 \int_{\exp(-mcs^2)}^{\exp(cs^2)} |y_1|^{-1} dy_1 \right)^m$$

$$\le C|s|^n(1+|s|)^m((m+1)cs^2)^m \approx C'(1+|s|^{n+3m}).$$

$\triangle$

Final remarks

Our goal was to present the idea, so we used simple arguments even though we got weaker end result. If the estimates are done in a more involved way one may replace $n+3m$ in (1.1) by a smaller number (we checked that $(n+3m)/2$ is enough), however we expect that in (1.2) it is enough to have more than $n/2+m$ derivatives in L^2, and getting this requires new ideas. Also, constants in (1.2) grow exponentially with the diameter of support of F. We may get polynomial growth, but we would like to have a uniform bound on $\|F(tL)\|_{L^1}$.

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