Functional calculus for slowly decaying kernels

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1. Introduction.

Let Ω be a metric space with the metric d and a Borel measure μ. Let

$$B(x,r) = \{y \in \Omega : d(x,y) < r\}.$$

We assume that there are constants C, q (*volume growth constants* of Ω) such that

$$\mu(B(x,sr)) \leq C\mu(B(x,r))s^q$$

for all $s > 1$. The existence of such constants is equivalent to the doubling condition, but here we are interested in getting q as small as possible (at the cost of enlarging C).

The purpose of the paper is to present a reasonably general approach to functional calculus, multipliers and almost everywhere convergence theorems on Ω. We consider integral operators with kernels decaying polynomialy away from diagonal. Our methods are based on L^2 estimates obtained from spectral theorem and careful use of weight functions.

The main question is to find sufficient conditions on positive definite operator A on Ω and on function F such that some of the following holds:

$$F(A) \qquad \text{is bounded on } L^1$$

$$F(A) \qquad \text{is bounded on } L^p,\ 1 < p < \infty \text{ and of weak type (1,1).}$$

$$F^*(A)f = \sup_{t>0} |F(tA)f| \qquad \text{is bounded on } L^p,\ 1 < p < \infty \text{ and of weak type (1,1).}$$

The subject has long history. Let us only mention works [15], [5], [17], [21], [18]. Our topic is closely related to works on Riesz means, for example [16], [23], [3].

In a few classical examples our results are weaker then those previously known. However, it seems that proofs of, say, multiplier theorems go as follows. One splits the multiplier into diadically supported pieces. For each piece one gets estimates for dominant term via some kind of Plancherel formula, which requires a lot of specific knowledge - unavailable in our setting. Then there is error term estimate - in many cases long and very technical. Final step is use of covering or decomposition arguments to get estimates for the whole

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multiplier. Our methods seem well suited to handle error term estimates while for the main term we use essentially a trivial estimate. However, if better estimates for main term are known, we can use them. It is quite possible that in our general setting the trivial estimate is the best one. We also give an improved way of gluing estimates for pieces to get full multiplier theorem — unlike the case of singular integrals we need no smoothness assumptions (cancelations are provided by the L^2 theory).

The main factor affecting our estimates is the volume growth rate. When the work on this paper begun this was believed to be the correct factor. Recent works [11], [20], [12], [14] shows that the picture is much more complicated.

2.Banach Algebras

Definition: A continuous function $\varphi : \Omega \times \Omega \to R$ is called submultiplicative if for all x,y,z

$$\varphi(x, y) \geq 1$$

and

$$\varphi(x, y)\varphi(y, z) \geq \varphi(x, z).$$

Of course if a,$b > 0$, then $\omega_a = (1 + d)^a$, e^{bd}, $\omega_a e^{bd}$ are submultiplicative. Our functional calculus is based on Banach $*$-algebras whose elements are kernels K, $K(x, y)$ being a complex number. For a submultiplicative function φ we write

$$\|K\|_{B(\varphi)} = \sup_y \int |K(x, y)|\varphi(y, x)d\mu(x)$$

$$K^*(x, y) = \bar{K}(y, x)$$

$$|K|_{B(\varphi)} = \max\{\|K\|_{B(\varphi)}, \|K^*\|_{B(\varphi)}\}$$

and we define the Banach $*$-algebra with unit element by

$$B(\varphi) = \{K : |K|_{B(\varphi)} < +\infty\} + CI.$$

The multiplication is defined by

$$K_1K_2(x, y) = \int K_1(x, s)K_2(s, y)d\mu(s).$$

Obviously

$$\int |K_1K_2(x, y)|\varphi(x, y)d\mu(x)$$

$$\leq \int\int |K_1(x, s)K_2(s, y)|d\mu(s)\varphi(x, y)d\mu(x)$$

$$\le \int\int |K_1(x,s)|\varphi(x,s)|K_2(s,y)|\varphi(s,y)d\mu(s)d\mu(x)$$
$$\le \int \|K_1\|_{B(\varphi)}|K_2(s,y)|\varphi(s,y)d\mu(s)$$
$$\le \|K_1\|_{B(\varphi)}\|K_2\|_{B(\varphi)}$$

for every x. Hence

$$\|K_1K_2\|_{B(\varphi)} \le \|K_1\|_{B(\varphi)}\|K_2\|_{B(\varphi)}$$

and

$$|K_1K_2|_{B(\varphi)} \le |K_1|_{B(\varphi)}|K_2|_{B(\varphi)}.$$

For a given kernel K we are going to estimate the norm of the element Ke^{inK} in $B(\omega_a)$. We will use the abbreviations

$$\|K\|_a = \|K\|_{B(\omega_a)},$$
$$|K|_a = |K|_{B(\omega_a)}.$$

(2.1). **Theorem**. *Let $a > b \ge 0$ and Ω be as above. For every L there exists constant M depending only on a, b, L and volume growth constants C, q but independent of Ω, d, μ, K such that if*

$$K = K^*,$$
$$\|K\|_a \le L,$$
$$\sup_x \mu(B(x,1)) \int |K(x,y)|^2 d\mu(y) \le 1,$$

then for every n

$$|e^{inK}|_b \le M(1+|n|)^\kappa$$

where $\kappa = 2^{[(b+q/2)/(a-b)]}((b+q/2)(1+1/(a-b))+1)$.

Proof: Put $A = \exp(iK)$, then $\|A\|_a = |A|_a \le \exp(L)$. Let $p = q/2+b$, $\delta = a-b$, $r \ge (n\|A\|_a)^{1/\delta}$ (r will be chosen later). Then $|A|_a r^{-\delta} \le 1/n$.

We put

$$A_0(x,y) = \begin{cases} A(x,y) & \text{for } d(x,y) < er \\ 0 & \text{otherwise,} \end{cases}$$

and

$$A_k(x,y) = \begin{cases} A(x,y) & \text{for } re^k \le d(x,y) < re^{k+1} \\ 0 & \text{otherwise.} \end{cases}$$

If $k \ge 1$,

$$\|A_k\|_b \le (re^k)^{b-a}\|A\|_a = C_1 e^{-k\delta}$$

where $C_1 = r^{-\delta}\|A\|_a$. Also

$$\|A_0\|_{L^2\to L^2} \le 1 + \|A - A_0\|_{L^2\to L^2} \le 1 + r^{-\delta}|A|_a = C_0 \le 1 + 1/n.$$

(2.2). **Lemma**. *Let A_i be as above, α be a multiindex, E a kernel such that*

$$\sup_y \mu(B(y,1))\|E(\cdot,y)\|_{L^2}^2 \le M^2$$

and $E(x,y) = 0$ for $d(x,y) > r_0$. Then

$$\|\prod_{i=1}^{|\alpha|} A_{\alpha(i)} E\|_b \le MC(r_0 + r\sum e^{\alpha(i)})^p \exp(-\delta \sum \alpha(i)) C_1^{|\{i:\alpha(i)>0\}|} C_0^{|\alpha|}$$

and

$$|\prod_{i=1}^{|\alpha|} A_{\alpha(i)}|_b \le C|\alpha| \left(r\sum e^{\alpha(i)}\right)^p \exp(-\delta \sum \alpha(i)) C_1^{|\{i:\alpha(i)>0\}|} C_0^{|\alpha|}.$$

Proof:

$$\|\prod_{i=1}^{|\alpha|} A_{\alpha(i)}\|_{L^2\to L^2} \le \prod_{i=1}^{|\alpha|}\|A_{\alpha(i)}\|_{L^2\to L^2} \le \exp(-\delta\sum\alpha(i)) C_1^{|\{i:\alpha(i)>0\}|} C_0^{|\alpha|}$$

Fix y. Put $U = \{x : d(x,y) \le r_0 + er\sum e^{\alpha(i)}\}$. We have

$$\|\prod_{i=1}^{|\alpha|} A_{\alpha(i)} E(\cdot,y)(1+d(\cdot,y))^b\|_{L^1}$$

$$= \|\chi_U \prod_{i=1}^{|\alpha|} A_{\alpha(i)} E(\cdot,y)(1+d(\cdot,y))^b\|_{L^1}$$

$$\le \sup_{x\in U}(1+d(x,y))^b \|\chi_U\|_{L^2} \|\prod_{i=1}^{|\alpha|} A_{\alpha(i)} E(\cdot,y)\|_{L^2}$$

$$\le (1 + r_0 + er\sum e^{\alpha(i)})^b |B(r_0 + er\sum e^{\alpha(i)}, y)|^{1/2}$$

$$\times \|\prod_{i=1}^{|\alpha|} A_{\alpha(i)}\|_{L^2\to L^2} \|E(\cdot,y)\|_{L^2}$$

$$\le C(r_0 + r\sum e^{\alpha(i)})^p \exp(-\delta\sum\alpha(i)) C_1^{|\{i:\alpha(i)>0\}|} C_0^{|\alpha|} M$$

which gives the first assertion. We prove the second assertion inductively. If $\alpha(|\alpha|) > 0$,

$$\|\prod_{i=1}^{|\alpha|} A_{\alpha(i)}\|_b \le \|\prod_{i=1}^{|\alpha|-1} A_{\alpha(i)}\|_b \|A_{\alpha(|\alpha|)}\|_b$$

and we estimate the first factor by the inductive assumption. If $\alpha(|\alpha|)=0$,

$$A_0 = I + A_0'$$

where

$$\sup_y \mu(B(y,1))\|A_0'(\cdot,y)\|_{L^2} \le \sup_y \mu(B(y,1))\|(A-I)(\cdot,y)\|_{L^2} \le C$$

so

$$\|\prod_{i=1}^{|\alpha|-1} A_{\alpha(i)}A_0\|_b \le \|\prod_{i=1}^{|\alpha|-1} A_{\alpha(i)}\|_b + \|\prod_{i=1}^{|\alpha|-1} A_{\alpha(i)}A_0'\|_b$$

and we estimate the first term by the inductive assumption and the second by the first assertion of the lemma. Applying * to all the operators in the proof we get not only estimate for $\|\cdot\|_b$ but also for $|\cdot|_b$.

Let us note the following obvious conseqence of (2.2) .

(2.3). **Lemma**. *If* $|\alpha| \le n$, *then*

$$|\prod_{i=1}^{|\alpha|} A_{\alpha(i)}|_b \le C|\alpha|^{p+1}\left(re^{\max\alpha(i)}\right)^p \exp(-\delta\sum\alpha(i))C_1^{|\{i:\alpha(i)>0\}|}.$$

(2.4). **Lemma**. *For every* $\varepsilon>0$ *there exists constant* C *such that for every* b, Ω, $f: N\mapsto R$, $n\in N$ *and kernel* A *on* Ω *if*

$$\|\prod_{i=1}^{n} A_{\alpha(i)}\|_b \le f(n)\exp(-\varepsilon\sum\alpha(i))(1/n)^{|\{i:\alpha(i)>0\}|}$$

then

$$|A^n|_b \le Cf(n).$$

Proof:

$$\begin{aligned}|A^n|_b = \|A^n\|_b &\le \sum_{|\alpha|=n}\|\prod_{i=1}^{n} A_{\alpha(i)}\|_b \\ &\le f(n)\sum_{|\alpha|=n}\exp(-\varepsilon\sum\alpha(i))(1/n)^{|\{i:\alpha(i)>0\}|} \\ &\le f(n)(1+1/n\sum_{k=1}^{\infty}e^{-\varepsilon k})^n \\ &\le f(n)\exp(\sum_{k=1}^{\infty}e^{-\varepsilon k}) = Cf(n).\end{aligned}$$

We expect main part of our kernels to live in the distance of order nr to the diagonal. In this region we sometimes have another estimate, and the following lemma allows us to use it efficiently.

(2.5). **Lemma**. *For every $\varepsilon > 0$ there exists constant C such that for every b, Ω, $f : N \mapsto R$, $n \in N$ and kernels A and E on Ω if $nC_1 \leq 1$, $E(x,y) = 0$ for $d(x,y) > nr$ and*

$$\|\prod_{i=1}^{n} A_{\alpha(i)} E\|_b \leq f(n) \exp(-\varepsilon \sum \alpha(i)) C_1^{|\{i:\alpha(i)>0\}|}$$

then

$$\sup_{y\in\Omega} \int_{d(x,y)>4nr} |A^n E(x,y)|(1 + d(x,y))^b d\mu(x) \leq C f(n) n C_1.$$

Proof: As $4nr > enr + nr$ we have

$$\sup_{y\in\Omega} \int_{d(x,y)>4nr} |A^n E(x,y)|(1 + d(x,y))^b d\mu(x) \leq \sum_{\substack{|\alpha|=n \\ \alpha\neq 0}} \|\prod_{i=1}^{n} A_{\alpha(i)} E\|_b$$

$$\leq f(n) \sum_{\substack{|\alpha|=n \\ \alpha\neq 0}} C_1^{|\{i:\alpha(i)>0\}|} \exp(-\varepsilon \sum \alpha(i))$$

$$\leq f(n) n C_1 \sum_{|\alpha|=n} \exp(-\varepsilon \sum \alpha(i))(1/n)^{|\{i:\alpha(i)>0\}|}$$

$$\leq C f(n) n C_1.$$

(2.6). **Lemma**. *Let m be a natural number. For every n and every sequence $(a_i)_{i=1}^n$ of nonnegative real numbers there exists subset I of $[1,n] \cap N$ such that $|I| \leq 2^{m-1} - 1$ and for every subset J of $[1,n] \cap N$ such that $|J| \leq |I| + 1$ and $J \cap I = \emptyset$ we have*

$$m \sum_{j\in J} a_j \leq \sum_{i=1}^{n} a_i$$

Proof: Without any loss of generality we assume that the sequence is nonincreasing and long enough, otherwise we renumerate it and add zeros.

Next

$$\sum a_i \geq \sum_{k=0}^{m-1} \sum_{2^k}^{2^{k+1}-1} a_i$$

so there exists $k \leq m-1$ such that

$$\sum a_i \geq m \sum_{2^k}^{2^{k+1}-1} a_i.$$

We take $I = [1, 2^k - 1] \cap N$. Of course the first assertion holds. Let J be as in the second assertion. Since (a_i) is nonincreasing

$$m \sum_{j \in J} a_j \leq m \sum_{2^k}^{2^{k+1}-1} a_i$$

which ends the proof.

(2.7). **Lemma**. *Let m be a positive natural number, A and Ω be as above. There exists a constant C such that for all n*

$$\| \prod_{i=1}^{n} A_{\alpha(i)} \|_b \leq C r^{(2^{m-1})p} n^{(2^{m-1})(p+1)} \exp((\frac{p}{m} - \delta) \sum \alpha(i)) C_1^{|\{i : \alpha(i) > 0\}|}$$

Proof: Fix α. Take $a_i = \alpha(i)$ in Lemma (2.6) and choose subset I according to the Lemma. We write $l = |I|$ and $I = \{i_j : j = 1, \ldots, l\}$. Put $i_0 = 0$ and $i_{l+1} = n+1$. By the Lemma $l \leq 2^{m-1} - 1$ and

$$m \sum_{j=1}^{l+1} \max_{i_{j-1} < i < i_j} \alpha(i) \leq \sum \alpha(i).$$

Also, by (2.3) ,

$$\| \prod_{i=i_{j-1}+1}^{i_j - 1} A_{\alpha(i)} \|_b$$

$$\leq C n^{p+1} \left(r \exp(\max_{i_{j-1} < i < i_j} \alpha(i)) \right)^p \exp(-\delta \sum_{i_{j-1} < i < i_j} \alpha(i)) C_1^{|\{i : i_{j-1} < i < i_j, \alpha(i) > 0\}|}$$

Next

$$\| \prod_{i=1}^{n} A_{\alpha(i)} \|_b \leq \| \prod_{i=1}^{i_1 - 1} A_{\alpha(i)} \|_b \prod_{j=1}^{l} \left(\left\| A_{\alpha(i_j)} \right\|_b \left\| \prod_{i=i_j+1}^{i_{j+1}-1} A_{\alpha(i)} \right\|_b \right)$$

$$\leq C (n^{p+1} r^p)^{2^{m-1}} \exp(p \sum_{j=1}^{l+1} \max_{i_{j-1} < i < i_j} \alpha(i)) \exp(-\delta \sum_i \alpha(i)) C_1^{|\{\alpha(i) > 0\}|}.$$

and the lemma follows.

End of the proof of (2.1) . We take $m = [p/\delta] + 1$ and $r = (n \|A\|_a)^{1/\delta}$. Then (2.1) holds by (2.7) and (2.4) .

(2.8). **Theorem**. *Let $a > b$, K, Ω satisfy assumptions of (2.1) and*

$$s > 2^{[(b+q/2)/(a-b)]}((b + q/2)(1 + 1/(a - b)) + 1) + 1/2.$$

There exists C such that for all $f \in H(s)$

$$|f(K)|_b \le C\|f\|_{H(s)}.$$

Proof: We may assume that $\|K\|_{L^2} \le 1$ (if not we replace K by $K/\|K\|_{L^2}$ and adjust f). Therefore $\operatorname{sp} K \subset [-1, 1]$ and $f(K)$ does not depend on the values of f outside $[-1, 1]$. Putting $h = \phi f$ where $\phi \in C_c^\infty([-2, 2])$ and $\phi = 1$ on $[-1, 1]$ we have $f(K) = h(K)$. Next

$$h(K) = \sum \hat{h}(n)e^{inK}$$

and

$$|h(K)|_b \le \sum |\hat{h}(n)||e^{inK}|_b \le M \sum |\hat{h}(n)|(1 + |n|)^\kappa \le \sum |\hat{h}(n)|(1 + |n|)^s(1 + |n|)^{\kappa-s}$$
$$\le M\|h\|_{H(s)}(\sum(1 + |n|)^{2(\kappa-s)})^{1/2} \le C\|f\|_{H(s)}$$

where κ and M are as in (2.1) .

(2.9). **Lemma**. *Let I and J be closed intervals such that $I \subset \operatorname{Int} J \subset J \subset (-\pi, \pi)$. Let $B(s)$ be $C(s)$ or $H(s)$, $f \in B(s_0)$, $\operatorname{supp} f \subset I$, $l > 0$. There exist functions f_j, $j = 0, 1, \ldots$ satisfying the following conditions*

$$f = \sum f_j$$
$$\operatorname{supp} f_j \subset J$$
$$|\hat{f}_j(k)| \le C(s_0, I, J, l)2^{-s_0 j}(1 + \max(0, |k| - 2^j))^{-l-3}\|f\|_{B(s_0)}$$
$$\|f_j\|_{B(0)} \le C(s_0, I, J, l)2^{-s_0 j}\|f\|_{B(s_0)}$$

where $C(s_0, I, J, l)$ depend only on s_0, I, J, l.

We choose smooth functions φ, ψ such that $\operatorname{supp} \varphi \subset J$, $\varphi|_I = 1$, $\psi = 1$ on $[-\frac{1}{2}, \frac{1}{2}]$, $\operatorname{supp} \psi \subset [-1, 1]$ and we put

$$h_j(x) = \begin{cases} \sum \psi(k)\hat{f}(k)e^{ikx} & \text{for } j = 0, \\ \sum[\psi(2^{-j}k) - \psi(2^{-j+1}k)]\hat{f}(k)e^{ikx} & \text{otherwise,} \end{cases}$$
$$f_j = \varphi h_j$$

Third condition holds because

$$\hat{f}_j(k) = \sum_r \hat{\varphi}(r)\hat{h}_j(k - r)$$

and $|\hat{\varphi}(k)| \le C(1+|k|)^{-l-4}$.

Remark. (2.9) is valid for very general scales of Banach spaces. In particular it is valid for Besov and Tribel-Lizorkin spaces.

(2.10). **Theorem**. *For every $b \ge 0$, $s > (b+q/2)(1+2/(a-b))$, $a > 2b+q/2$ there exist C such that*

$$|f(K)K|_b \le C\|f\|_{C(s)}.$$

Proof: Without any loss of generality we assume that $\operatorname{supp} f \subset (-\pi,\pi)$. We decompose f as in (2.9) (with $l > s$). Fix k. We put $n = 2^{k+2}$, $r = (n^2\|A\|_a)^{(1/(a-b))}$, $p = b+q/2$,

$$E(x,y) = \begin{cases} K(x,y) & \text{for } d(x,y) < nr \\ 0 & \text{otherwise.} \end{cases}$$

Observe that

$$\int\limits_{d(x,y)>4nr} |f_k(K)K(x,y)|(1+d(x,y))^b d\mu(y)$$

$$\le \int\limits_{d(x,y)>4nr} |f_k(K)E(x,y)|(1+d(x,y))^b d\mu(y)$$

$$+\|f_k(K)\|_b \int\limits_{d(x,y)>nr} |K(x,y)|(1+d(x,y))^b d\mu(y) = I_1 + \|f_k(K)\|_b I_2$$

As $I_2 \le Cn^{-2}$ and $p < \delta$, we get estimate for the second term as in (2.8) . Also

$$I_1 \le \sum_{|j|\le 2^{k+2}} |\hat{f}_k(j)| \int\limits_{d(x,y)>4nr} |A^j E(x,y)|(1+d(x,y))^b d\mu(y) + \sum_{|j|>2^{k+2}} |\hat{f}_k(j)|\, \|A^j\|_b \|K\|_b$$

and by (2.2) , (2.5) and (2.9) (note that $C_1 \le \frac{1}{n^2}$), this is

$$\le Cn^{-s}(nr)^p$$

Next, putting $U = \{x : d(x,y) \le 4nr\}$, we have

$$\int\limits_U |f_k(K)K(x,y)|(1+d(x,y))^b d\mu(x)$$

$$\le \|\chi_U\|_{L^2} \sup_{x\in U}(1+d(x,y))^b \|f_k(K)K(\cdot,y)\|_{L^2}$$

$$\le \|f_k\|_{L^\infty} \sup_{x\in U}(1+d(x,y))^b \|\chi_U\|_{L^2}\|K(\cdot,y)\|_{L^2}$$

$$\le Cn^{-s}(nr)^p.$$

Gathering the estimates above we get

$$\|f_k(K)K\|_b \le Cn^{-s}(nr)^p \le C'2^{-k\varepsilon}$$

for some $\varepsilon > 0$, which of course implies our claim.

(2.11). **Theorem**. *Let I and J be closed intervals such that $I \subset \mathrm{Int}\, J \subset J$. Assume $a > 2b+q/2$. If there exist κ, γ, χ such that $\kappa \ge 1$, $b \ge 0$, $s > (b+q/2)(1+\kappa/(a-b))+2-\kappa$, $s > (b+\gamma/2)(1+\kappa/(a-b))$, $s > \chi + b(1+\kappa/(a-b))$ and for every f such that* $\mathrm{supp}\, f \subset J$

$$\int\limits_{d(x,y)<r} |f(K)|(x,y)dx \le M(r^{\gamma/2}\|f\|_{L^2} + \|f\|_{H(\chi)}),$$

then for every f such that $\mathrm{supp}\, f \subset I$

$$\|f(K)K\|_b \le M'\|f\|_{H(s)}.$$

Proof:The proof is similar to the proof of (2.10) . We choose $r = (n^\kappa\|A\|_a)^{(1/(a-b))}$ to get $C_1 = n^{-\kappa}$, also l in (2.9) is made large enough. Then we use the assumption to estimate integral over the set U:

$$\int\limits_U |f_k(K)K(x,y)|(1+d(x,y))^b d\mu(x)$$

$$\le (1+4(nr)^b)\int\limits_U |f_k(K)K(x,y)|d\mu(x)$$

$$\le M(1+4(nr)^b)((nr)^{\gamma/2}\|xf_k\|_{L^2} + \|xf_k\|_{H(\chi)})$$

$$\le C(nr)^{b+\gamma/2}n^{-s} + C_1(nr)^b\|f_k\|_{H(\chi+b(1+\kappa/(a-b)))}^{\chi/(\chi+b(1+\kappa/(a-b)))}\|f_k\|_{L^2}^{b(1+\kappa/(a-b))/(\chi+b(1+\kappa/(a-b)))}$$

$$\le C(nr)^{b+\gamma/2}n^{-s} + C_2(nr)^b(n^{-s})^{b(1+\kappa/(a-b))/(\chi+b(1+\kappa/(a-b)))} \le C_3 2^{-k\varepsilon}$$

3. Multiplier theorems.

Let A be a non-negative self-adjoint densely defined operator on $L^2(M,\mu)$. Let E be the spectral measure of A. By the spectral theorem, we write

$$Af = \int \lambda dE(\lambda)f$$

and

$$e^{-tA}f = \int e^{-\lambda}dE(\lambda)f.$$

We assume that

$$e^{-tA}f(x) = \int e^{-tA}(x,y)f(y)d\mu(y)$$

where the kernels $e^{-tA}(x,y)$ satisfy the following estimates :

there exist positive numbers a, m, α and C such that for all t

$$\sup_y \int |e^{-tA}(x,y)|(1+t^{-1/m}d(x,y))^a d\mu(x) \le C$$

$$\sup_y \mu(B(y,t^{1/m})) \int |e^{-tA}(x,y)|^2 d\mu(x) \le C$$

(3.1). **Theorem**. *Assume that the conditions above are satisfied and that*

$$\sup_{y,z} \int |e^{-tA}(x,y) - e^{-tA}(x,z)| d\mu(x) \le Ct^{-\alpha/m} d(y,z)^\alpha.$$

If $F \in H(s)_{loc}$,

$$s > 2^{[q/(2a)]}((q/2)(1+1/a)+1)+1/2$$

and for a non-zero $\varphi \in C_c^\infty(R_+)$

$$\sup_{t>0} \|F_t\varphi\|_{H(s)} \le M$$

where

$$F_t(\lambda) = F(t\lambda),$$

then $F(A) = \int F(\lambda)dE(\lambda)$ *is of weak type (1,1) and bounded on* $L^p(M)$, $1 < p < \infty$.

(3.2). **Lemma**. *If for some constants* $R < 1$, M, $a > 0$, $\alpha > 0$, *a family of kernels* $\{K_n\}_{n=0}^\infty$ *satisfies*

$$\|K_n\|_{B((1+R^n d)^a)} \le M$$

$$\int |K_n(x,y) - K_n(x,z)| d\mu(x) \le MR^{n\alpha} d(y,z)^\alpha,$$

then for some C *depending only of* R, a, α

$$\sup_{z,y} \int\limits_{d(x,y)>2d(y,z)} \sum_n |K_n(x,y) - K_n(x,z)| d\mu(x) \le MC$$

Proof: Fix y and z. Let

$$S = \{x : d(x,y) > 2d(y,z)\}.$$

Now

$$\int\limits_S |K_n(x,y) - K_n(x,z)| d\mu(x)$$

$$\le \int\limits_S |K_n(x,y)| d\mu(x) + \int\limits_S |K_n(x,z)| d\mu(x) \le 2MR^{-na} d(y,z)^{-a}.$$

Then

$$\sum \int_S |K_n(x,y) - K_n(x,z)| d\mu(x)$$

$$\leq \sum \min(MR^{n\alpha} d(y,z)^{\alpha}, 2MR^{-na} d(y,z)^{-a})$$

$$\leq 2M(1-R^a)^{-1} + M(1-R^{\alpha})^{-1} \leq CM.$$

Proof of (3.1) : First note that if $\mu(M) = \infty$ then second assumption about e^{-tA} implies that the spectral measure of A has no atom at 0. If $\mu(M) < \infty$ then this assumption implies that the spectral projector corresponding to 0 is bounded on L^1 (beeing bounded from L^1 to $L^2 \subset L^1$) and hence by interpolation and duality on all L^p, $1 \leq p \leq \infty$. In any case we need to handle only (strictly) positive part of the spectrum. Choose $\varphi \in C_c^{\infty}(R_+)$ such that $\sum \varphi(2^{mn}\lambda) = 1$ for $\lambda > 0$. Let

$$F_n(\lambda) = \varphi(\lambda) F(2^{-mn}\lambda),$$

$$G_n(\lambda) = F_n(-\log(\lambda)),$$

$$H_n(\lambda) = F_n(-\log(\lambda))\lambda^{-1},$$

$$e_n = e^{-2^{mn}A},$$

$$K_n = G_n(e_n) = H_n(e_n)e_n.$$

We have

$$F(A) = \sum F_n(2^{mn}A) = \sum K_n.$$

By the assumption,

$$\sup_y \int |e_n(x,y)|(1 + 2^{-n} d(x,y))^a d\mu(x) \leq C$$

$$\sup_y \mu(B(y, 2^n)) \int |e_n(x,y)|^2 d\mu(x) \leq C$$

and of course,

$$\|G_n\|_{H(s)} \leq C'$$

therefore replacing d by $2^{-n}d$ we may apply (2.8) to get

$$\|K_n\|_{B((1+2^{-n}d)^{\varepsilon})} \leq M$$

for sufficiently small $\varepsilon > 0$. Moreover

$$|K_n(x,y) - K_n(x,z)| \leq \int |H_n(e_n)(x,s)||e_n(s,y) - e_n(s,z)| d\mu(s)$$

hence

$$\int |K_n(x,y) - K_n(x,z)| d\mu(x) \leq \|H_n(e_n)\|_0 \int |e_n(s,y) - e_n(s,z)| d\mu(s) \leq M 2^{-\alpha n} d(y,z)^\alpha$$

This means that assumptions of (3.2) are satisfied.

Now, we use the general theory of Calderón-Zygmund operators (see for example Coifman-Weiss [4]).

(3.3). **Theorem**. *If $K = \sum K_k$, K is bounded on L^p, $p > 1$, $K_k \Psi_j = 0$ for $k > j$, $K_k \Psi_j = K_k$ for $k < j$ and*

$$\int |K_k(x,y)|(1 + 2^{-k} d(x,y))^\epsilon \leq C$$

$$|\Psi_k(x,y)| \leq C(1 + 2^{-k} d(x,y))^{-q-\epsilon}(B(x,2^k)^{-1} + B(y,2^k)^{-1})$$

then K is of weak type 1-1.

Put

$$\rho_t(x,y) = (1 + t^{-1} d(x,y))^{-q-\varepsilon}(\mu(B(x,t))^{-1} + \mu(B(y,t))^{-1}),$$

$$\tilde{M} f(x) = \sup_{t>0} \int \rho_t(x,y)|f(y)| dy.$$

(3.4). **Lemma**.

$$\tilde{M} f(x) \leq C M f(x).$$

In particular $\tilde{M}$ is bounded on L^p, for all $1 < p \leq \infty$.

Proof:There exists C such that

$$\mu(B(x, 2d(x,y))) \leq C(1 + t^{-1} d(x,y))^q \mu(B(x,t)),$$

$$\mu(B(x, 2d(x,y))) \leq \mu(B(y, 4d(x,y))) \leq C(1 + t^{-1} d(x,y))^q \mu(B(y,t))$$

for all x, y, $t > 0$. Hence

$$\rho_t(x,y) \leq C'(\mu(B(x, \max(t, 2d(x,y)))))^{-1}(1 + t^{-1} d(x,y))^{-\varepsilon}.$$

Fix t. We have

$$\int \rho_t(x,y)|f(y)| dy \leq C'' \sum_{k=0}^{\infty} 2^{-\varepsilon} \mu(B(x, 2^k t))^{-1} \int_{B(x,2^k t)} |f(y)| dy$$

$$\leq C''' M f(x).$$

If f is an arbitrary L^1 function and λ is a positive real number, then either $\mu(M) \leq \|f\|_{L^1}/\lambda$ and

$$\mu(\{x : |Kf(x)| > \lambda\}) \leq \mu(M) \leq \|f\|_{L^1}/\lambda$$

or we apply to f the Calderon-Zygmund decomposition at height λ (see for example Coifman-Weiss [4], Chapitre 3, Theoreme (2.2)) and obtain balls $B(x_i, r_i)$ such that putting $P = \bigcup B(x_i, r_i)$ we have

$$\|f\big|_{M-P}\|_{L^\infty} \leq C\lambda$$

$$\int\limits_{B(x_i,r_i)} |f| \leq C\lambda\mu(B(x_i, r_i))$$

$$\sum \mu(B(x_i, r_i)) \leq C\|f\|_{L^1}/\lambda$$

and every $x \in M$ belongs to at most C different $B(x_i, r_i)$.

Put $Q_i = B(x_i, r_i)$, $Q_i^* = B(x_i, 2r_i)$, $S_i = Q_i - \cup_{j<i} Q_j$, $f_i = \chi_{S_i} f$, $k_i = [\log_2(r_i)]$.

(3.5). **Lemma**. *There exists C such that for all i*

$$\sum_n \int\limits_{(Q_i^*)^c} |K_n(1 - \Psi_{k_i})f_i|(x)dx \leq C\|f_i\|_{L^1}.$$

Proof:

$$\int\limits_{(Q_i^*)^c} |K_n f_i|(x)dx \leq \|f_i\|_{L^1} \sup_{y \in Q_i} \int\limits_{(Q_i^*)^c} |K_n(x, y)|dx$$

$$\leq \|f_i\|_{L^1} \sup_y \int\limits_{d(x,y)>2^{k_i-1}} |K_n(x, y)|dx$$

$$\leq 2^{\epsilon(n-k_i+1)}\|f_i\|_{L^1} \sup_y \int\limits_{d(x,y)>2^{k_i-1}} |K_n(x, y)|(1 + 2^{-n}d(x, y))^\epsilon dx$$

$$\leq C2^{\epsilon(n-k_i)}\|f_i\|_{L^1}$$

By assumption, $K_n(1 - \Psi_{k_i}) = 0$ for $n > k_i$ and $K_n(1 - \Psi_{k_i}) = K_n$ for $n < k_i$. Hence

$$\sum_n \int\limits_{(Q_i^*)^c} |K_n(1 - \Psi_{k_i})f_i|(x)dx \leq \|K_{k_i}(1 - \Psi_{k_i})f_i\|_{L^1} + \sum_{n<k_i} \int\limits_{(Q_i^*)^c} |K_n f_i|(x)dx$$

$$\leq C\|f_i\|_{L^1} \sum_{n \leq k_i} 2^{\epsilon(n-k_i)}$$

$$\leq C\|f_i\|_{L^1}.$$

(3.6). **Lemma**. *For every $1 \leq p < \infty$ there exist C_p such that*

$$\|\sum \Psi_{k_i} f_i\|_{L^p}^p \leq C_p\lambda^{p-1}\|f\|_{L^1}.$$

Proof:Put $\tau_i = \rho_{2^{k_i}}$. It is easy to check (using doubling condition) that

$$\sup_{y\in Q_i} \tau_i(x,y) \le C \inf_{y\in Q_i} \tau_i(x,y)$$

with C uniform in x and i. Fix i and x and choose $y_0 \in Q_i$.

$$|\Psi_{k_i} f_i| \le \int \tau_i |f_i|(y) d\mu(y)$$

$$\le C\lambda\mu(Q_i)\tau_i(x,y_0)$$

$$\le C'\lambda\tau_i\chi_{Q_i}.$$

Let $r = p/(p-1)$. If $h \in L^r$, $h \ge 0$, then

$$|(h, \tau_i\chi_{Q_i})| = |(\tau_i h, \chi_{Q_i})| \le (\tilde{M}h, \chi_{Q_i}).$$

By (3.4) ,

$$(h, \sum |\Psi_{k_i} f_i|) \le C(\tilde{M}h, \sum \lambda\chi_{Q_i}) \le C\|h\|_{L^r}\|\sum \lambda\chi_{Q_i}\|_{L^p}$$

By properties of Calderón-Zygmund decomposition $\|\sum\lambda\chi_{Q_i}\|^p_{L^p} \le C\sum\lambda^p\mu(Q_i) \le C'\lambda^{p-1}\|f\|_{L^1}$, which ends the proof of (3.6) .

Let

$$g = f - \sum_i (1-\Psi_{k_i})f_i = f - \sum_i f_i + \sum_i \Psi_{k_i} f_i$$

so

$$Kf = K(\sum_i (1-\Psi_{k_i})f_i + g) = \sum_n\sum_i K_n(1-\Psi_{k_i})f_i + Kg$$

We have

$$\|g\|^p_{L^p} \le p(\|f - \sum_i f_i\|^p_{L^p} + \|\sum_i \Psi_{k_i} f_i\|^p_{L^p})$$

$$\le p(\|f - \sum_i f_i\|^{p-1}_{L^\infty}\|f - \sum_i f_i\|_{L^1} + C\lambda^{p-1}\|f\|_{L^1}) \le C'\lambda^{p-1}\|f\|_{L^1}.$$

Put $E = \cup_i Q_i^*$. By (3.5) ,

$$\int_{M-E} |\sum_n\sum_i K_n(1-\Psi_{k_i})f_i|(x)dx \le \sum_n\sum_i \int_{(Q_i^*)^c} |K_n(1-\Psi_{k_i})f_i|(y)dy$$

$$\le \sum_i C\|f_i\|_{L^1} \le C'\|f\|_{L^1}.$$

Finally

$$|\{|Kf| > \lambda\}| \le |\{|Kg| > \lambda/2\}| + |\{|\sum_n\sum_i K_n(1-\Psi_{k_i})f_i| > \lambda/2\}|$$

$$\leq \frac{(2\|Kg\|_{L^p})^p}{\lambda^p} + |E| + \frac{2\int\limits_{M-E} |\sum_n \sum_i K_n(1-\Psi_{k_i})f_i|(x)dx}{\lambda}$$

$$\leq C(\frac{\lambda^{p-1}\|f\|_{L^1}}{\lambda^p} + \frac{\|f\|_{L^1}}{\lambda}) \leq 2C\lambda^{-1}\|f\|_{L^1}$$

which ends the proof of (3.3) .

(3.7). **Theorem**. *If $a > q$, $l > (1+2/a)q/2+1$, $p > 1$ and for a non-zero $\varphi \in C_c^\infty(R_+)$,*

$$\int_0^\infty t^{-1}\|\varphi\delta(t)F\|_{W_l^p}dt < \infty,$$

then

$$F^*f = \sup_{t>0} |\delta(t)F(A)f|$$

is of weak type (1,1) and bounded on L^p for $1 < p \leq \infty$.

Remark. The index l in (3.7) can not be essentially lowered. This contrasts with homogeneous multipliers on R^n, (see [22]) where one can take l close to $(n+1)/2$ (or use derivatives in L^2 and l close to $n/2$). To see this let A be operator on $C_c^\infty(R^2)$ defined by the formula

$$\hat{Af} = \psi\hat{f}$$

where $\hat{}$ denotes the Fourier transform. Assume that ψ is homogeneous with respect to anisotropic dilations, that is $\psi(tx, t^2y) = t^k\psi(x, y)$ and that the level set $\{(x, y) : \psi(x, y) = 1\}$ contains an interval I not parallel to any of the coordinate axes. Let supp $\hat{f}$ be contained in small neighbourhood of a point x in I. Take $F(\lambda) = (1-\lambda)^{(1-\epsilon)}$. Then

$$(F(A)f)(x, y) \approx (x^2 + y^2)^{-1+\epsilon/2}$$

when (x, y) lies in a bounded distance from direction normal to I and $F(A)f$ is small outside this set. If we take $F(tA)f$ then we get similar estimate but the set where $F(tA)f$ is large changes. More precise, it is obtained applying anisotropic dilation to the set where $F(A)f$ is large. Anisotropic dilations act transitively on directions different from axes, so

$$F^*(A)f(x, y) \approx (x^2 + y^2)^{-1+\epsilon/2}$$

on a cone with nonempty interior. But then clearly $F^*(A)$ is not of weak type 1-1. If k is large then A satisfies our estimates with large a. One can modify this example to get differential A, (then level set must be tangent of high order to I).

(3.8). **Theorem**. *If $a > q$, $s > (1+2/a)q/2$ and for a non-zero $\varphi \in C_c^\infty(R_+)$ and a* *constant* C

$$\sup_{t>0} \|\varphi\delta(t)F\|_{C(s)} \le C,$$

then $F(A)$ *is of weak type (1, 1) and bounded on* L^p *for* $1 < p < \infty$.

Remark. With other assumptions as in (3.8) , if

$$\int_0^\infty t^{-1}\|\varphi\delta(t)F\|_{C(s)} dt < \infty,$$

then $F(A)$ is bounded on L^1.

Remark. (3.8) applied to the sublaplacean on Lie group of polynomial growth gives result essentially equivalent to that of G. Aleksopoulos [1].

(3.9). **Lemma**. *There exists* C *such that for all* x, y *and* t

$$|e^{tA}(x,y)| \le C\mu(B(y,t^{1/m}))^{-1}.$$

Proof. We are going to prove that for all $\varepsilon > q/2$ there exists C such that

$$|e^{tA}(x,y)| \le C(1+t^{-1/m}d(x,y))^{-(a-\varepsilon)}(\mu(B(x,t^{1/m}))\mu(B(y,t^{1/m})))^{-1/2}$$

for all x, y and t. This easily implies our claim.

The estimate above is trivially true for $\varepsilon = a$. We have (using Schwartz inequality)

$$|e^{2tA}(x,y)|(1+t^{-1/m}d(x,y))^{(a-\varepsilon)}(\mu(B(x,t^{1/m}))\mu(B(y,t^{1/m})))^{1/2} =$$

$$\int |e^{tA}(x,s)e^{tA}(s,y)|(\mu(B(x,t^{1/m}))\mu(B(y,t^{1/m})))^{1/2}(1+t^{-1/m}d(x,y))^{(a-\varepsilon)}ds$$

$$\le \sup_y \mu(B(y,t^{1/m})) \int |e^{tA}(s,y)|^2(1+t^{-1/m}d(s,y))^{2(a-\varepsilon)}ds$$

$$\le \sup_y \int |e^{tA}(s,y)|(1+t^{-1/m}d(s,y))^a ds$$

$$\times \sup_{s,y} \mu(B(y,t^{1/m}))|e^{tA}(s,y)|(1+t^{-1/m}d(x,y))^{(a-2\varepsilon)}$$

$$\le C \sup_{s,y}(\mu(B(x,t^{1/m}))\mu(B(y,t^{1/m})))^{1/2}|e^{tA}(s,y)|(1+t^{-1/m}d(x,y))^{(a-2\varepsilon+q/2)}$$

Repeating this we can get $\varepsilon - q/2$ arbitrarily small and thus (3.9) is proved.

We fix φ and ψ such that φ, ψ are in $C^\infty(R)$, $\operatorname{supp}\varphi \subset [2^{-m}, 2^{m/2}]$, $(\forall x > 0)$ $\sum \varphi(2^{mk}x) = 1$, and $\operatorname{supp}\psi \subset [-1,1]$, with $\psi(x) = 1$ for $x \in [0, 2^{-m/2}]$. Let

$$\varphi_k(\lambda) = \varphi(2^{mk}\lambda),$$

$$\psi_k(\lambda) = \psi(2^{mk}\lambda),$$
$$\Psi_k = \psi_k(A),$$
$$F_k(\lambda) = \varphi_k(\lambda)F(\lambda),$$
$$G_k(\lambda) = F_k(-2^{-mk}\log(\lambda))\lambda^{-1},$$
$$e_k = e^{-2^{mk}A},$$
$$W_k = \psi_k(-2^{-mk}\log(\lambda))\lambda^{-2},$$
$$K_k = G_k(e_k)e_k.$$

We have
$$F(A) = \sum F_k(A) = \sum G_k(e_k)e_k = \sum K_k.$$
By the assumption,
$$\sup_y \int |e_k(x,y)|(1+2^{-k}d(x,y))^a d\mu(x) \le C$$
$$\sup_y \mu(B(y,2^k)) \int |e_k(x,y)|^2 d\mu(x) \le C$$
and of course,
$$\|G_k\|_{C(l)} \le C'$$
therefore replacing d by $2^{-k}d$ we may apply (2.10) to get
$$\|K_k\|_{B((1+2^{-k}d)^{\varepsilon})} \le M$$
for sufficiently small $\varepsilon > 0$.

Also
$$\psi_k(A) = W_k(e_k)e_k^2$$
and for any s
$$\|W_k\|_{H(s)} \le C_s$$
therefore replacing d by $2^{-k}d$ we may apply (2.8) to get
$$\int |W_k(e_k)|(x,y)(1+2^{-k}d(x,y))^{a-\epsilon} \le C$$
By (3.9) (and symmetry) we get
$$\sup_y |W_k(e_k)e_k|(x,y) \le \mu(B(x,2^k))^{-1}$$
Next
$$|\psi_k(A)|(x,y) \le \int |W_k(e_k)e_k|(x,s)|e_k|(s,y)ds$$

$$\leq \int\limits_{d(x,s)\geq d(x,y)/2} |W_k(e_k)e_k|(x,s)|e_k|(s,y)ds + \int\limits_{d(y,s)\geq d(x,y)/2} |W_k(e_k)e_k|(x,s)|e_k|(s,y)ds$$

$$\leq \sup_s |e_k|(s,y) \int\limits_{d(x,s)\geq d(x,y)/2} |W_k(e_k)e_k|(x,s)ds$$

$$+ \sup_s |W_k(e_k)e_k|(x,s) \int\limits_{d(y,s)\geq d(x,y)/2} |e_k|(s,y)ds$$

$$\leq C(1+t^{-1/m}d(x,y))^{-(a-\varepsilon)}(\mu(B(x,t^{1/m}))^{-1}+\mu(B(y,t^{1/m}))^{-1})$$

In other words

$$|\psi_k(A)|(x,y) \leq C(1+2^{-k}d(x,y))^{-(a-\varepsilon)}(\mu(B(x,2^{-k}))^{-1}+\mu(B(y,2^{-k}))^{-1})$$

which is the second assumption in (3.3) . This ends the proof of (3.8) .

The following lemma is all what is needed to end the proof of (3.7) .

(3.10). **Lemma**.

$$\int \sup_{t>0} |F(tA)\varphi_k(A)|(x,y)(1+2^{-k}d(x,y))^{\epsilon}dx \leq C\int_0^\infty t^{-1}\|\varphi\delta(t)F\|_{W_l^p}dt < \infty.$$

Proof:We have

$$\int \sup_{t>0} |F(tA)\varphi_k(A)|(x,y)(1+2^{-k}d(x,y))^{\epsilon}dx$$

$$\leq \sum_j \int \sup_{t>0} |F_j(tA)\varphi_k(A)|(x,y)(1+2^{-k}d(x,y))^{\epsilon}dx$$

and

$$\sum_j \|\delta(2^{-mj})F_j\|_{W_l^p} \leq C\int_0^\infty t^{-1}\|\varphi\delta(t)F\|_{W_l^p}dt$$

so it is enough to show that

$$\int \sup_{t>0} |F_j(tA)\varphi_k(A)|(x,y)(1+2^{-k}d(x,y))^{\epsilon}dx \leq C\|\delta(2^{-mj})F_j\|_{W_l^p}.$$

Without any loss of generality we may assume that $k=0$ and $j=0$. Indeed, otherwise we replace d by $2^{-k}d$ and F by $\delta(2^{-mj})F$. We write

$$e_n = \exp(ine^{-A})e^{-A},$$

$$H_t(\lambda) = (\varphi_0)(-\log(\lambda))F_0(-t\log(\lambda))/\lambda.$$

We have

$$F_0(tA)\varphi(A) = H_t(e^{-A})e^{-A} = \sum_n \hat{H}_t(n)e_n$$

and $H_t = 0$ for $t \notin [2^{-2m}, 2^{2m}]$. It follows

$$\sup_{t>0} |\varphi(A)F_0(tA)|(x,y) \le \sum_n \sup_{2^{2m} \ge t > 2^{-2m}} |\hat{H}_t(k)||e_n(x,y)|$$

and

$$\int \sup_{t>0} |F_0(tA)\varphi(A)|(x,y)(1+2^{-k}d(x,y))^\epsilon dx$$

$$\le \sum_n \sup_{2^{2m} \ge t > 2^{-2m}} |\hat{H}_t(n)| \int |e_n(x,y)|(1+2^{-k}d(x,y))^\epsilon dx$$

$$\le C\|F_0\|_{W_l^p} \sum_n (1+|n|)^{-l}\|e_n\|_\epsilon$$

$$\le C'\|F_0\|_{W_l^p} \sum_n (1+|n|)^{-l}(1+|n|)^{(1+2/(a-\epsilon))q/2} \le C''\|F_0\|_{W_l^p}.$$

Let M be a homogeneous group (cf. [7]) of homogeneous dimension Q, that is a Lie group equipped with one-parameter family $\{\delta_t\}_{t>0}$ of automorphisms such that for all $x \in M$

$$\lim_{t\to 0} \delta_t x = e$$

where e is the neutral element of M and for every compact set $A \subset M$

$$\mu(\delta_t(A)) = t^Q \mu(A)$$

where μ is the Haar measure. Such a Q is not unique — as additional normalization we require all eigenvalues of $D_e\delta_t$ to be at most t for $t < 1$ and to have t as one of the eigenvalues. We assume that A is a left-invariant hypoelliptic differential operator, homogeneous of degree $m > 0$, that is (for f in the domain of A)

$$A(f \circ \delta_t) = t^m (Af) \circ \delta_t.$$

Such an A is usually called Rockland operator. Since A is left-invariant we have

$$F(A)f = f * H_F$$

where H_F is a distribution on M called the kernel of $F(A)$. In our setting we have a kind of Plancherel formula:

$$\|H_F\|_{L^2} = c\int |F(A)|^2(x)x^{-1+Q/m}dx.$$

To see this note that as $\|H_F\|_{L^2} = H_{|F|^2}(e)$ the formula above is equivalent to

$$H_F(e) = c\int F(A)(x)x^{-1+Q/m}dx.$$

For $F(x) = e^{-x}$ both sides are finite (and nonzero) so one can choose c to have equality. Then homogeneity shows that the set of F for which equality holds is closed under dilations. Thus the equality holds for linear combinations of exponentials - that is on a set dense in $L^1(x^{-1+Q/m}dx)$, which shows the formula. We fix a riemmanian metric d on G. Note that it can happen that q (which describes growth of balls for riemmanian metric) is smaller then Q.

(3.11). **Theorem**. *If A and G are as above, $s > q/2$ and for a non-zero $\varphi \in C_c^\infty(R_+)$ and a constant C*

$$\sup_{t>0} \|\varphi\delta(t)F\|_{H(s)} \le C,$$

then $F(A)$ is of weak type (1, 1) and bounded on L^p for $1 < p < \infty$.

Remark. If A is a homogeneous sublaplacean this theorem reduces to the theorem of M. Christ [2] and G. Mauceri and S. Meda [19]. If A is nondifferential then possible values of a are bounded and we must assume that $a > q$ and $s > (1 + 2/a)q/2$. J. Dziubański [6] showed how to apply results like (3.11) for some nondifferential A when a is small to get $s = q/2 + \epsilon$ with arbitrarily small $\epsilon > 0$.

Proof:First choose a so that $a > Q$ (hence $a > q$), $s > (1 + 2/a)q/2$. Then we proceed as in the proof of (3.8) . The difference is that instead of $2^{-k}d$ we use $d_k(x, y) = d(\delta(2^{-k})x, \delta(2^{-k})y)$ where d is some fixed left-invariant riemmanian metric on G. Also thanks to the Plancherel formula we may apply (2.11) instead of (2.10) . Indeed, we should estimate $\|K_k\|_{B((1+d_k)^\varepsilon)}$ (for Ψ_k we directly re-use the proof of (3.8)). Using dilations, we reduce to problem to the estimate for $\|K_0\|_{B((1+d)^\varepsilon)}$. We write $\tilde{f}(x) = f(e^{-x})$ so $f(e_0) = \tilde{f}(A)$ (and we assume that f have support in some fixed interval I contained in positive reals — we want to be away from 0). Then

$$\int_{d(x,e)<r} |f(e_0)|(x, e)dx \le |\{x : d(x, e) < r\}|^{1/2}\|f(e_0)\|_{L^2}$$

$$\le Cr^{q/2}\|\tilde{f}(A)\|_{L^2} = C_1 r^{q/2}(\int |\tilde{f}(x)|^2 x^{-1+Q/m}dx)^{1/2}$$

$$\le C_2 r^{q/2}\|\tilde{f}\|_{L^2} \le C_3 r^{q/2}\|f\|_{L^2}.$$

Since our kernels are left-invariant the estimate above means that assumptinons of (2.11) are satisfied with $\kappa = 2$, $\gamma = q$, $\chi = 0$ so we get uniform bound on $\|K_k\|_{B((1+d_k)^\varepsilon)}$.

To finish the proof choose a homogeneous (dilation invariant) and left-invariant metric ρ on G (see for example [13]). We have

$$(1 + \rho(x, y)) \leq C(1 + d(x, y))$$

and because ρ is homogeneous

$$(1 + 2^{-k}\rho(x, y)) \leq C(1 + d_k(x, y)).$$

Hence assumptions of (3.3) are satisfied with d replaced by ρ which ends the proof.

Suppose that M is a smooth compact manifold of dimension q without boundary. Assume that A is an elliptic differential operator on M of order m which is positive definite on L^2 with respect to a smooth positive density $d\mu$. Fix a riemmanian metric d on M.

(3.12). **Lemma**. *If M and A are as above, $q > 1$, m is order of A,* $\operatorname{supp} f \subset [1/4, 4]$, *$t < 1$, then there is C such that*

$$\|f(tA)\|^2_{L^1,L^2} \leq Ct^{-q/m}(t^{q/m}\|f\|^2_{H(q/2)} + \|f\|^2_{L^2}).$$

Proof:We need Hörmander's estimate on spectral projections (see [16]) for $a > 1/4$:

$$\|E([a, a + a^{(m-1)/m}))\|^2_{L^1,L^2} \leq Ca^{(q-1)/m}.$$

Put $a = t^{-1}/4$, $h = t^{-(m-1)/m)}$, $n = 15([t^{-1/m}] + 1)$. Then, using orthogonality and Hörmander's estimate we get

$$\|f(tA)\|^2_{L^1,L^2} \leq \sum_{k=0}^{n} \|f(tA)E([a + kh, a + (k + 1)h))\|^2_{L^1,L^2}$$

$$\leq Ct^{-(q-1)/m} \sum_{k=0}^{n} \sup_{x \in [(a+kh)t,(a+(k+1)ht)} |f(x)|^2.$$

For any interval I we have

$$\sup_{x \in I} |f|^2 \leq C(|I|^{-1}\|f\|^2_{L^2(I)} + |I|\|\partial_x f\|^2_{L^2(I)})$$

so

$$\sum_{k=0}^{n} \sup_{x \in [(a+kh)t,(a+(k+1)ht)} |f(x)|^2 \leq C((ht)^{-1}\|f\|^2_{L^2} + (ht)\|\partial_x f\|^2_{L^2})$$

Also

$$\|\partial_x f\|^2_{L^2} \leq C\|f\|^{2(q-2)/q}_{L^2}\|f\|^{4/q}_{H(q/2)} \leq C'((ht)^{-2}\|f\|^2_{L^2} + (ht)^{q-2}\|f\|^2_{H(q/2)}).$$

Finally

$$\|f(tA)\|^2_{L^1,L^2} \le Ct^{-(q-1)/m}((ht)^{-1}\|f\|^2_{L^2} + (ht)^{q-1}\|f\|^2_{H(q/2)})$$
$$\le Ct^{-q/m}(\|f\|^2_{L^2} + t^{q/m}\|f\|^2_{H(q/2)}).$$

(3.13). **Theorem**. *If M is a compact riemmanian manifold, q is the dimension of M, A is an elliptic differential operator of order m, positive definite on L^2 with respect to a smooth measure μ, $F \in C(R)$, $s > q/2$ and for a non-zero $\varphi \in C_c^\infty(R_+)$ and a constant C*

$$\sup_{t>0} \|\varphi\delta(t)F\|_{H(s)} \le C,$$

then $F(A)$ is of weak type (1, 1) and bounded on L^p for $1 < p < \infty$.

Remark. A variant of (3.13) is valid on noncompact manifolds. One needs bounded geometry assumptions to have global bound in Hörmander's estimate and to control local behaviour of the semigroup. Also conclusion asserts only boundedness of operator with kernel restricted to a neighbourhood of diagonal.

Remark. Theorem remains valid form pseudodifferential operators. However, the a one gets depend on m so one must pass to the powers of A (like in [6]).

Proof:We give the proof only for $q > 1$. (If $q = 1$ one must replace $1/m$ in (3.12) by a larger number, and then tediously check that the proofs remain valid).

From the local Sobolev lemma we deduce that if $F \in C_c(R)$, then $F(A)$ has bounded kernel, hence is bounded on all L^p, $1 \le p \le \infty$. Therefore we can assume that $\operatorname{supp} F \subset [1, \infty)$. The semigroup generated by differential operator A satisfies our assumptions with any $a > 0$ as long as t is bounded (see for example [10]). Because of our assumption about support of F estimates for large t are not needed. We proceed as in the proof of (3.8) but using (3.12) and (2.11) instead of (2.10) . Indeed we should estimate $\|K_k\|_{B((1+d_k)^\varepsilon)}$ for $k < 0$. Fix k. Put $d_k = 2^{-k}d$. We write $\tilde{f}(x) = f(e^{-x})$ so $f(e_k) = f(e^{-2^{mk}A}) = \tilde{f}(2^{mk}A)$. Then

$$\int_{d_k(x,e)<r} |f(e_k)|(x,y)dx \le |\{x : d(x,e) < r2^k\}|^{1/2}\|f(e_k)(\cdot,y)\|_{L^2}$$
$$\le Cr^{q/2}2^{kq/2}\|\tilde{f}(2^{mk}A)\|_{L^1,L^2}$$
$$\le C_1 r^{q/2}2^{kq/2}2^{-kq/2}(2^{kq/2}\|\tilde{f}\|_{H(q/2)} + \|\tilde{f}\|_{L^2})$$
$$\le C_2 r^{q/2}(2^{kq/2}\|f\|_{H(q/2)} + \|f\|_{L^2})$$
$$\le C_3(r^{q/2}\|f\|_{L^2} + \|f\|_{H(q/2)}).$$

In the last line we use compactness of our manifold – we may assume that $r2^k$ is bounded. Thus, we checked that assumptions of (2.11) are satisfied with $\kappa = 2$, $\gamma = q$, $\chi = q/2$.

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