Spectral multipliers on exponential growth solvable Lie groups

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Introduction

Let M be a measure space and let L be a positive definite operator on $L^2(M)$. By the spectral theorem, for any bounded Borel measurable function $F : [0, \infty) \mapsto \mathbb{C}$ the operator $F(L)f = \int_0^\infty F(\lambda) dE(\lambda) f$ is bounded on $L^2(M)$.

We are interested in sufficient conditions on F for $F(L)$ to be bounded on $L^p(M)$, $p \neq 2$. We direct the reader to [1], [3], [4], [8], [9], [10], [12] and [13] for more background on various multiplier theorems.

In this paper we assume F is compactly supported and have some smoothness (finite number of derivatives) and we consider only the case $p = 1$. Our measure space G is semidirect product of stratified nilpotent Lie group N and the real line. The operator L is (minus) sublaplacian on G. Our group has exponential volume growth. The earlier theory suggested that one needs holomorphic F for $F(L)$ to be bounded on L^1, however the recent results [5], [6], [7] showed that estimates on only a finite number of derivatives of F imply boundedness of $F(L)$ on L^1 on some solvable G of exponential growth. In this case we say that G (more precisely L) has C^k-functional calculus. On the other hand, Christ and Müller give an example of a solvable Lie group on which F must be holomorphic. The problem is to find the condition on G (and possibly L) which decides whether G has a C^k-functional calculus or not. Here, our condition is in terms of roots of adjoint representation of the Lie algebra of G. Our groups are of "rank one", but unlike [7], we allow groups with commutant of arbitrarily large step of nilpotency.

Preliminaries

Let N be a stratified nilpotent Lie algebra of step q, that is,

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$$N = \bigoplus_{j=1}^{q} V_j \ ,$$

and $[V_j, V_i] \subset V_{i+j}$ for every $1 \leq j, i \leq q$. We assume that V_1 generates N.

A *dilation structure* on a stratified Lie algebra N is a one parameter group $\{e^{sD}\}$ of automorphisms of N determined by

$$DX = jX \qquad \text{for} \;\; X \in V_j.$$

If we consider N as a nilpotent Lie group with the multiplication given by the Campbell-Hausdorff formula

$$xy = x + y + \frac{1}{2}[x, y] + ...,$$

then $\{e^{sD}\}$ forms a group of automorphisms on the group N, and the nilpotent Lie group N equipped with the dilations $\{e^{sD}\}$ is said to be a *stratified homogeneous group.*

One easily checks that the Lebesgue measure on N is also biinvariant Haar measure.

There exists a number $Q > 0$ such that for all bounded measurable $F \subset N$

$$|e^{sD} F| = e^{sQ} |F|,$$

this Q is called the *homogeneous dimension* of N. It is evident that

$$Q = \operatorname{tr}(D) = \sum_{j=1}^{q} j \cdot \dim V_j.$$

We choose and fix a *homogeneous norm* on N, that is, a continuous, positive, symmetric, and smooth away from 0 function $x \mapsto |x|$ which vanishes only for $x = 0$, and satisfies $|e^{sD} x| = e^s |x|$.

Let $G = \mathbb{R} \times N$, with the multiplication given by the formula

$$(u_1, n_1)(u_2, n_2) = (u_1 + u_2, e^{-u_2 D} n_1 n_2).$$

Then

$$(u, n)^{-1} = (-u, e^{uD} n^{-1}).$$

Let a weight function w be defined as $w(u, n) = |n|^Q$. Note, that the Lebesgue measure is left invariant, modular function δ is given by

$$\delta(u, n) = e^{uQ},$$

and we have

$$\int f(g)dg = \int \delta(g) f(g^{-1})dg,$$

$$\int f(g)w(g)dg = \int \delta^2(g) f(g^{-1})w(g)dg.$$

Assume $X_1, \ldots, X_m$ generate N and are of order 1 (that is $\mathrm{lin}\{X_1, \ldots, X_m\} = V_1$). We will identify X_j with left invariant vector fields on G. We denote by $\tilde{X}_j$ the corresponding right invariant vector fields on G. We put $\tilde{X}_0 = \partial_u$. Right invariant vector fields generate *left* translations so

$$\langle \tilde{X}_j f, f\rangle = -\langle f, \tilde{X}_j f\rangle \qquad \text{for } j = 0, \ldots, m.$$

We have $\delta|_N = 1$ so

$$\langle X_j f, f\rangle = -\langle f, X_j f\rangle \qquad \text{for } j = 1, \ldots, m.$$

We write

$$L = \sum_{j=0}^{m} \tilde{X}_j^2.$$

The heat kernel p_t is defined by the formula $e^{tL} f = p_t * f$. In the sequel we will identify convolution operators with kernels (functions on G). The real operators in the algebra generated by L are self-adjoint, which in terms of kernel reads:

$$F(L)(g) = \delta(g) F(L)(g^{-1}).$$

Note that the formula is valid for complex F.

Let d be (right) invariant riemannian metric on G. There is a constant C such that

$$B_r = \{(u, n) : d((u, n), 0) < r\} \subset$$

$$\{(u, n) : |u| < Cr, |n| < C(e^{Cr} + 1)\}.$$

A straightforward calculation shows that for some C

$$\int_{d(g,0)<r} (1 + w(g))^{-1} dg \leq Cr^2.$$

Results

(1.1). **Theorem**. *There exists C such that for every $s \in \mathbb{R}$ we have*

$$\|p_{1+is}\|_{L^1(G)} \le C(1+|s|^{\frac{Q+4}{2}}).$$

(1.2). **Theorem**. *For every compactly supported $F \in C^{\left[\frac{Q+7}{2}\right]}$ (or F in the Sobolev space $H^{(\frac{Q+5}{2}+\epsilon)}$) the operator $F(-L)$ is bounded on $L^1(G)$*

Theorem (1.2) is a consequence of (1.1) . Indeed, using the spectral theorem and the inversion formula for the Fourier transform we have

$$F(-L) = \frac{1}{2\pi}\int \hat{f}(s) p_{1-is} ds,$$

where $f(x) = F(x)e^x$. $F \in H^{(\frac{Q+5}{2}+\epsilon)}$ implies that $\int |\hat{f}(s)|(1+|s|)^{\frac{Q+4}{2}} ds$ is convergent.

We are going to prove (1.1) . If $Q = 1$, then G is affine goup of real line and the results follows [5]. In the sequel we assume $Q \ge 2$. We have, (by [7] Lemma (1.4))

$$\int |p_{1+is}(g)|^2 e^{Md(g,0)} dg \le C \exp(C(1+s^2)M^2),$$

also, there is R such that

$$\int e^{-Rd(g,0)} dg < \infty$$

so

$$\int |p_{1+is}(g)| e^{d(g,0)} dg$$

$$\le \left(\int |p_{1+is}(g)|^2 e^{(R+1)d(g,0)} dg\right)^{1/2} \left(\int e^{-Rd(g,0)} dg\right)^{1/2}$$

$$\le C \exp(Cs^2).$$

Consequently, if $r = Cs^2$

$$\int |p_{1+is}(g)| dg \le \int\limits_{d(g,0)<r} |p_{1+is}(g)| dg + \int\limits_{d(g,0)\ge r} |p_{1+is}(g)| dg$$

$$\le \int\limits_{d(g,0)<r} |p_{1+is}(g)(1+w(g))^{1/2}|(1+w(g))^{-1/2} dg$$

$$+ e^{-r} \int\limits_{d(g,0)\ge r} |p_{1+is}(g)| e^{d(g,0)} dg$$

$$\leq \|p_{1+is}(1+w)^{1/2}\|_{L^2}\|(1+w)^{-1/2}\|_{L^2(B_r)} + e^{-r}\int |p_{1+is}(g)|e^{d(g,0)}dg$$

$$\leq C|s|^2\|p_{1+is}(1+w)^{1/2}\|_{L^2} + e^{-Cs^2}Ce^{Cs^2}.$$

So to finish the proof we need to know that

$$\|w^{1/2}p_{1+is}\| \leq C(1+|s|)^{Q/2}.$$

We write

$$\phi(s) = \|w^{1/2}p_{\frac{1}{2}+(i+\alpha)s}\|^2$$

and

$$f = p_{\frac{1}{2}+(i+\alpha)s}.$$

We have

$$\partial_s\phi(s) = 2\Re\langle(i+\alpha)Lf, wf\rangle$$

$$= 2\sum_{j=0}^{m}\Re\langle(i+\alpha)\tilde{X}_j^2 f, wf\rangle.$$

For $j > 0$ we compute

$$\langle\tilde{X}_j^2 f, wf\rangle = \int(\tilde{X}_j^2 f)(g)\bar{f}(g)w(g)dg$$

$$= \int(\delta(g)(X_j^2 f)(g^{-1}))(\delta(g)\bar{f}(g^{-1}))w(g)dg$$

$$= \int \delta^2(g)(X_j^2 f)(g^{-1})\bar{f}(g^{-1})w(g)dg$$

$$= \int X_j^2 f(g)\bar{f}(g)w(g)dg = \langle X_j^2 f, wf\rangle.$$

Next

$$\langle X_j^2 f, wf\rangle = -\langle X_j f, wX_j f\rangle - \langle X_j f, (X_j w)f\rangle.$$

Because of the homogeneity $|X_j w| \leq C|w|^{(Q-1)/Q}$ so

$$\Re\langle(i+\alpha)\tilde{X}_j^2 f, wf\rangle = \Re(-(i+\alpha)\langle X_j f, wX_j f\rangle - (i+\alpha)\langle X_j f, (X_j w)f\rangle$$

$$\leq -\alpha\|w^{1/2}X_j f\|^2 + C\|w^{1/2}X_j f\|\|(X_j w)w^{-1/2}f\|$$

$$\leq -\alpha\|w^{1/2}X_j f\|^2 + \alpha\|w^{1/2}X_j f\|^2 + \frac{1}{\alpha}C'\|w^{\frac{Q-2}{2Q}}f\|^2 = \frac{1}{\alpha}C'\|w^{\frac{Q-2}{2Q}}f\|^2$$

$$\leq \frac{1}{\alpha}C'\|w^{1/2}f\|^{\frac{2Q-4}{Q}}\|f\|^{\frac{4}{Q}}.$$

For $j = 0$,

$$\Re\langle (i+\alpha)\tilde{X}_0^2 f, wf\rangle = -\alpha \|w^{1/2}\tilde{X}_0 f\|^2$$

Adding over j, we get

$$\partial_s \phi(s) \le C'' \frac{1}{\alpha} \|w^{1/2} f\|^{\frac{2Q-4}{Q}} \|f\|^{\frac{4}{Q}}$$
$$\le C''' \frac{1}{\alpha} \phi(s)^{\frac{Q-2}{Q}}.$$

Hence

$$\phi(s) \le (\phi(0)^{\frac{2}{Q}} + C\frac{s}{\alpha})^{\frac{Q}{2}}.$$

In terms of the semigroup (and putting $\alpha = \frac{1}{2s}$):

$$\|w^{1/2} p_{1+is}\| \le C(1+|s|)^{\frac{Q}{2}}.$$

Improvements and open problems

One can easily generalize (1.1) to a larger class of groups (and sublaplacians). In fact, what counts is that G is a semidirect product of nilpotent group and real line and that the matrix D has eigenvalues with positive real parts. Then one can build w making it homogeneous with respect to dilations generated by D.

Inspection of the proof of (1.1) shows that the crucial role is played by the inequality $|X_j w| \le C|w|^{(Q-1)/Q}$. However, this inequality may be replaced by weaker one: $|X_j w| \le C(|w|^q + 1)$ with $q < 1$, which is valid even when X_j are not eigenvectors of D.

Moreover, it is not necessary to assume that L is sum of squares of vector fields with distinguished field $\tilde{X}_0$ and other fields from Lie algebra of N. It is enough to assume that $L = \sum b_{j,k}\tilde{X}_j\tilde{X}_k$ with $b_{j,k}$ positive definite. Such an L may be written in our restricted form at the cost of changing the one dimensional subgroup complementing N.

The counterexample of M. Christ and D. Müller shows that one can not simply drop the assumption about eigenvalues. Finding correct conditions is an open problem. Also, while it is quite likely that theorem holds for AN groups with multidimensional A, new ideas are needed to handle those.

In the spirit of Hörmander multiplier theorem one would like the have

$$\|F(-tL)\|_{L^1} < C$$

for compactly supported F with C dependent only on finite number of derivatives of F (and independent of t). This is equivalent to the estimate

$$\|p_{l+is}\|_{L^1} < C(1+|s|/l)^M$$

with C and M independent of $l > 0$ and s. Our estimates gives much larger result for big l (and moderate $|s|/l$). It may be interesting to note that the exponent M we get seems to be far too big (on the Iwasawa AN groups with one dimensional A one has $M = 3/2$), and getting the correct one should also give the correct scaling (dependence on l).

Finally, let us mention that currently is not clear which properties of L are really needed. Significant part of our argument goes for larger classes of operators, however, we don't know whether (1.1) remains valid for Schrödinger operators or for some higher order operators (say sums of even powers of vector fields).

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