Hörmander type multiplier theorem on complex Iwasawa AN groups.

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Dedicated to the memory of Andrzej Hulanicki

Abstract

We prove that, for a distinguished laplacian on an Iwasawa AN group corresponding to complex semisimple Lie group, a Hörmander type multiplier theorem holds. Our argument is based on Littlewood-Paley theory.

1 Introduction and preliminaries

Multiplier theorems are a long studied subject. Most of works on multipliers of Hörmander type were done in polynomial growth setting. Operators on spaces of exponential growth are more difficult, in some cases (like laplacian on non-compact symmetric spaces) only holomorphic functions can give operators which are bounded on L^p, $p \neq 2$.

Currently, there are several known results (starting form [3] and [2]) on solvable groups of exponential growth. However only [4] and follow-up works give Hörmander type multiplier theorems, other works put additional restrictions on the multiplier so that at infinity the resulting operator is given by convolution with an integrable function. In [4] only distinguished laplacians on (a particular class of) groups of rank 1 are handled. The method of [4] can be extended to a distinguished laplacian on Iwasawa type solvable groups, however the full argument is long and only part of it is written up. This paper presents a different, much simpler argument for

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distinguished laplacian on Iwasawa type solvable groups corresponding to complex semisimple groups.

Let G denote a connected, complex semisimple Lie group and $\mathbf{g}$ its Lie algebra. Denote by θ a Cartan involution of $\mathbf{g}$, and write

$$\mathbf{g} = \mathbf{k} \oplus \mathbf{p}$$

for the associated Cartan decomposition. Fix a maximal abelian subspace $\mathbf{a}$ of $\mathbf{p}$; this determines a root space decomposition

$$\mathbf{g} = \mathbf{g}_0 \oplus \sum_{\alpha \in \Lambda} \mathbf{g}_\alpha,$$

Λ denoting the set of roots of the pair $(\mathbf{g}, \mathbf{a})$. Corresponding to a choice of the ordering of the roots, we have an Iwasawa decomposition

$$\mathbf{g} = \mathbf{a} \oplus \mathbf{n} \oplus \mathbf{k}.$$

Let $G = ANK$ be the corresponding Iwasawa decomposition of G. A distinguished laplacian on AN can be constructed as follows. Let $\pi : \mathbf{g} \to \mathbf{p}$ be the projection (defined by the Cartan decomposition). We define a positive definite form $\tilde{B}$ on $\mathbf{a} \oplus \mathbf{n}$ setting $\tilde{B}(X, Y) = B(\pi X, \pi Y)$ where B is the Killing form on $\mathbf{g}$. Put $n = \dim(AN)$. Choose an orthonormal (with respect to $\tilde{B}$) basis in $\mathbf{a} \oplus \mathbf{n}$, say $\{X_1, \ldots, X_n\}$.

Define (minus) the laplacian L by setting

$$Lf(x) = -\sum_{j=1}^{n} X_j^2.$$

where we identify elements of $\mathbf{a} \oplus \mathbf{n}$ with right-invariant vector fields on AN.

L is a densly defined selfadjoint operator on $L^2(AN)$ (integration is with respect to left-invariant Haar measure). For a bounded Borel measurable function m on $[0, \infty)$ we can define the bounded operator $m(L)$ on $L^2(AN)$ using the spectral theorem

$$m(L) = \int m(\lambda) dE(\lambda)$$

where E is the spectral measure of L. It is natural to ask for sufficient conditions on m which imply that $m(L)$ can be extended to $L^p(AN)$, $p \neq 2$.

2 Main theorem

Theorem 2.1 *If for some function $\psi \in C_c^\infty(R_+)$, $\psi \neq 0$,*

$$\sup_{t>0} \|\psi m(t\cdot)\|_{C^{n+1}} < \infty$$

then $m(L)$ is bounded on $L^p(AN)$, $1 < p < \infty$.

Remark. If the assumption in 2.1 is satisfied by one ψ, then it is satified by all ψ.

Let ϕ be a bounded holomorphic function defined for $\Re z > 0$ such that $|\phi(z)| \leq c(|z|/(1+|z|^2))$ and $\phi(x) > 0$ for positive real x. Put $\phi_k(\lambda) = \phi(2^{-k}\lambda)$. We define a vector valued operator S_ϕ by the formula:

$$S_\phi(f) = \{\phi_k(L)f\}_{k=-\infty}^{\infty}.$$

Fact 2.2 *S_ϕ is bounded from $L^p(dx)$ to $L^p(\ell^2)$.*

Proof: This is a consequence of the holomorphic multiplier theorem from [1] or [5], using classical arguments. $\diamond$

Choose $\psi \in C_c^\infty(\mathcal{R}_+)$ such that $\sum_k \psi(2^k x) = 1$ for all $x > 0$. Let $m_k(\lambda) = \psi(2^{-k}\lambda)m(\lambda)$ and $h_k = \phi_k^{-2} m_k$. Then

$$m(L) = \sum m_k(L) = \sum \phi_k(L)h_k(L)\phi_k(L) = S_\phi^* H S_\phi$$

where H is the bounded operator on $L^2(\ell^2)$ given by the formula:

$$H\{f_k\}_{k=-\infty}^{\infty} = \{h_k(L)f_k\}_{k=-\infty}^{\infty}$$

and $S_\phi^* : L^2(\ell^2) \mapsto L^2(dx)$ is the adjoint of S_ϕ.

Thus, to prove Theorem 2.1 we only need to prove that H is bounded on $L^p(dx, \ell^2)$.

Lemma 2.3 *There exists C such that for all k and f_k*

$$\|h_k(L)\|_{L^1,L^1} \leq C$$

$$|h_k(L)f_k|(x) \leq C \sup_{t>0} \exp(-tL)|f_k|(x)$$

Proof: For $x \in \mathcal{R}^n$ put $\eta_k(x) = h_k(2^k|x|^2)$. The functions η_k are in C_c^{n+1} with uniform bounds on their support and their derivatives so

$$|\widehat{\eta_k}|(y) \leq C_1(1+|y|)^{-n-1}$$

where $\widehat{\ }$ denotes the Fourier transform and C_1 does not depend on k. Next, there is a nonnegative integrable function w such that

$$C_1(1+|y|)^{-n-1} \leq \int_0^\infty w(t)\widehat{e_t}(y)dt$$

where $e_t(x) = \exp(-t|x|^2)$. For example, we can take multiple of $(1+t)^{-3/2}$ as w. Consequently,

$$|h_k(-\Delta)|(x) \leq C_1 \int_0^\infty w(t)\exp(t2^{-k}\Delta)(x)dt$$

where Δ is laplacian on $\mathcal{R}^n$. By [3] the last inequality remains valid on AN:

$$|h_k(L)|(x) \leq C_1 \int_0^\infty w(t) \exp(-t2^{-k}L)(x)dt$$

Since $\| \exp(t2^{-k}L)\|_{L^1,L^1} = 1$ the first claim follows. To get the second claim we note that

$$|h_k(L)f_k|(x) \leq C_1 \int_0^\infty w(t) \exp(-t2^{-k}L)|f_k|(x)dt$$

$$\leq C_1 \int_0^\infty w(t)dt \sup_{t>0} \exp(-t2^{-k}L)|f_k|(x) = C_2 \sup_{t>0} \exp(-tL)|f_k|(x).$$

◇

Now

$$\sup_k |h_k(L)f_k|(x) \leq C \sup_{t>0} \exp(-tL)(\sup_k |f_k|)(x)$$

Since the semigroup maximal function is bounded on L^p, H is bounded on $L^p(dx, \ell^\infty)$. Next, since

$$\|h_k(L)\|_{L^1(dx),L^1(dx)} \leq C$$

we have

$$\|H\{f_k\}_{k=-\infty}^\infty\|_{L^1(dx,\ell^1)} = \| \sum_k |h_k(L)f_k| \|_{L^1(dx)} = \sum_k \|h_k(L)f_k\|_{L^1(dx)}$$

$$\leq C \sum_k \|f_k\|_{L^1(dx)} = \|\{f_k\}_{k=-\infty}^\infty\|_{L^1(dx,\ell^1)}.$$

By analytic interpolation between $L^1(dx, \ell^1)$ and $L^2(dx, \ell^2)$ H is bounded on $L^p(dx, \ell^p)$, $1 \leq p \leq 2$. Again, by interpolation between $L^p(dx, \ell^p)$ and $L^p(dx, \ell^\infty)$ H is bounded on $L^p(dx, \ell^2)$, $1 \leq p \leq 2$. We handle $p > 2$ by duality, which ends the proof.

3 Possible improvements and limitations

In Lemma 2.3 it is enough to bound the Sobolev norm $H((n + 1)/2 + \varepsilon)$ of h_n. Since this is the only place where we use regularity of m, the main theorem remain valid if m only satisfies:

$$\sup_{t>0} \|\psi m(t\cdot)\|_{H(s)} < \infty.$$

with $s > (n + 1)/2$.

Lemma 2.3 (with n replaced by apropriate values like in [2]) remains valid for distinguished laplacian on all (not necessarily complex) Iwasawa AN groups, however the proof is much more complicated.

Since our argument is based on use of maximal function it probably cannot be improved to give expected critical exponent $n/2$. Also, it is probably impossible to get the weak type $(1,1)$ of the multiplier operator preserving the simplicity of the argument.

Finally, let us mention that the related problem of bounding Riesz transforms requires estimates of derivatives of the semigroup kernel, hence a quite different method.

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