On random walks on discrete solvable groups

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Introduction

This paper is concerned with the decay of $p_n(e)$ for symmetric random walk on discrete solvable group G of exponential growth. Previous results in [1], [3], [4], [6], [7], [8] shows that on Lie groups one has polynomial decay or decay of type $\exp(-n^{1/3})$ and that this remains true for polycyclic groups. Morover there is rather general estimate from above of the similar type. So one may conjecture that such a type of decay is correct for all solvable groups of exponential growth. We are going to show that in fact on discrete solvable groups much faster decay is possible. One may also notice that our groups have no finite presentation. As only groups with finite presentations can occur in geometric problems it would be nice to handle those, however we leave this problem open.

Note Our results seem to be a particular case of results from [5]. While [5] covers more topics and the technical details are different the idea for getting both lower and upper bound is the same. Morover, in [5] it is pointed out that one may modify the example to get finitely presented solvable groups with "fast" decay of $p_n(e)$.

Results

Let G_d be the semidirect product of Z^d and Z^{Z^d} and p_n be the simple random walk on G_d. We will call the Z^d part H. Then an element of G is a pair (h, f) with $h \in H$ and f beeing integer valued function on H taking only finitely many values different from 0. In the sequel we will denote by F the set of all such f. H acts on F by translations (we define translations by the formula $(T(h)f)(x) = f(xh)$). Multiplication on G is given by the formula

$$(h_1, f_1)(h_2, f_2) = (h_1h_2, T(h_2^{-1})f_1 + f_2).$$

Let A_h be a subgroup of G consisting of pairs $(0, f)$ with $f(x) = 0$ for $x \neq h$. Of course A_h is isomorphic to Z. Note that $T(h)^{-1}A_e = A_h$. We choose $(\delta_k, 0)$, $k = 1, \ldots, d$ and $(0, \delta_0)$ as generators of G_d. Then, the probability distribution p_1 is given by

$$p_1((\pm\delta_k, 0)) = p_1((0, \pm\delta_0)) = \frac{1}{2(d+1)}.$$

As usual, p_n is the n-th convolution power of p_1.

(1.1). **Theorem**. *For every d there exist C and $c > 0$ such that for all n*

$$c\exp(-Cn^{d/(d+2)}\log n) \le p_{2n}(e) \le C\exp(-cn^{d/(d+2)}).$$

There exist G and p_1 such that for every $\epsilon > 0$ $p_n(e)$ decays faster then $\exp(-n^{1-\epsilon})$.

(1.2). **Theorem**. *Let Z and Y be random variables with values in A_e and H respectively. Assume that Z and Y are symmetric, have finite second moment, $P(Z=0) > 0$, and that the values of Z (Y resp.) generate A_e (H resp.). Let Z_j and Y_j be independent random variables, Z_j have the same distribution as Z and Y_j have the same distribution as Y. Then there is $c > 0$ such that*

$$c\exp(-c^{-1}\log(n)n^{d/d+2}) \le \sup_{g\in G} P(\prod_{j=1}^{n}(Y_jZ_j) = g) \le \exp(-cn^{d/d+2}).$$

P. Note that $Y_jZ_j = (Y_j, Z_j)$ and

$$\prod_{j=1}^{n}(Y_j, Z_j) = (\prod_{j=1}^{n} Y_j, \prod_{j=1}^{n} T^{-1}(\prod_{l=j+1}^{n} Y_j)Z_j)$$

$$\sup_{g\in G} P(\prod_{j=1}^{n}(Y_jZ_j) = g) \le \sup_{f\in F} P(\prod_{j=1}^{n} T^{-1}(\prod_{l=j+1}^{n} Y_l)Z_j = f).$$

Since F is commutative

(1)
$$\prod_{j=1}^{n} T^{-1}(\prod_{l=j+1}^{n} Y_l)Z_j = \prod_{h\in H} T^{-1}(h) \prod_{h=\prod_{l=j+1}^{n} Y_l} Z_j$$

Since $T^{-1}(h)Z_j$ takes values in A_h, the first product above is simply a cartesian produkt (of independent variables). Morover, on each axis we may apply to Z_j the central limit theorem, so

$$P(\prod_{j=1}^{n} T^{-1}(\prod_{l=j+1}^{n} Y_l)Z_j = f) = E(\prod_{h\in H} P(\prod_{j=1}^{n_h} Z_j = f(h)))$$

$$\le E(\prod_{h\in H}(1+cn_h)^{-1/2}) \le E(\exp(-\tilde{c}|\{h\in H : n_h > 0\}|))$$

where n_h is the number of j such that $h = \prod_{l=j+1}^n Y_l$. By [2]

$$E(\exp(-\tilde{c}|\{h \in H : n_h > 0\}|)) \le \exp(-cn^{d/(d+2)}).$$

Similarly

$$P(\prod_{j=1}^n (Y_j, Z_j) \in H) = P(\prod_{j=1}^n T^{-1}(\prod_{l=j+1}^n Y_l)Z_j = 0) \ge E(\prod_{h \in H}(1 + cn_h)^{-1/2})$$

$$\ge E(\exp(-c_0 \log(2+n)|\{h \in H : n_h > 0\}|))$$

$$\ge E(\exp(-c_0|\{h \in H : n_h > 0\}|))^{\log(2+n)}$$

$$\ge c\exp(-c_1 n^{d/(d+2)})^{\log(2+n)}$$

where the last inequality follows from [2]. Next, since Y is symmetric $E(Y) = 0$ and $E(|\prod_{j=1}^n Y_j|^2) \le Cn$. Put

$$\phi(x) = c\exp(-c_1 x^{d/(d+2)})^{\log(2+x)}$$

nad let r be the smallest such that $Cnr^{-2} \le 1/2\phi(n)$. Then $r \le \frac{C_3}{\phi(n)}$ and

$$P(|\prod_{j=1}^n Y_j| > r) \le 1/2\phi(n)$$

so with probability at least $1/2\phi(n)$ the product $\prod_{j=1}^n (Y_j, Z_j)$ takes a value in the ball centered at 0 and of radius r in H. This ball has $C_4 r^d$ elements, so the maximal probability is at least

$$1/2\phi(n)/(C_4 r^d) \ge C_5 \phi(n)^{d+1}$$

which differs from ϕ only because of constants.

P. of (1.1) : Let X be a random variable with distributon p_1 and let Y nad Z be (independent) random variables such that

$$P(Z = (0, \pm\delta_0)) = 1/2 \qquad P(Y = (\pm\delta_k, 0) = 1/2d \quad k = 1, \ldots, d.$$

We consider X as a mixture of random variables Y and Z, that is X equals Y with probability $d/(d+1)$, otherwise X equals Z. Next, let X_i (or Y_i or Z_i) be independent random variables with the same distribution as X (resp. Y or Z). Consequently, we treat $\prod_{i=1}^n X_i$ as a product of Y_i and Z_i (with random choice between Y_i and Z_i). We group Y-s and Z-s into series. More precisely, consider sequence of independent (also from Y-s and Z-s) random variables ω such that $P(\omega(i)=0)=d/(d+1)$ and $P(\omega(i)=1)=1/(d+1)$. We have $X_i=Y_i$ iff $\omega(i)=0$ (and $X_i=Z_i$ iff $\omega(i)=1$). Let $A=\{\omega(1)=0\}$ and $B=\{\omega(1)=1\}$. Let m_i and l_i be defined inductively: $m_0=l_0=1$, $m_{i+1}=\min\{k : k> m_i, \omega(k-1)=0, \omega(k)=1\}$, $l_{i+1}=\min\{k : k>l_i, \omega(k-1)=1, \omega(k)=0\}$. The differences m_i-m_{i-1} are independent, and except for $i=1$ have a common distribution (which is convolution of two geometric distrbutions). There exists $c_1>0$ such that $P(l_{[c_1n]}>n)<\exp(-c_1n)$. On A put $\tilde{Y}_i=\prod_{k=l_{i-1}}^{m_i-1} Y_k$ and $\tilde{Z}_i=\prod_{k=m_i}^{l_i-1} Z_k$, on B put $\tilde{Y}_i=\prod_{k=l_i}^{m_i-1} Y_k$ and $\tilde{Z}_i=\prod_{k=m_i}^{l_{i+1}-1}$. Let $A_n=A\cap\{l_{[c_1n]}\leq n\}$. Then, on A_n

$$\prod_{i=1}^n X_i=\prod_{i=1}^{[c_1n]}(\tilde{Y}_i\tilde{Z}_i)\prod_{i=l_{[c_1n]}}^n X_i.$$

Let us note that $\tilde{Y}_i$ and $\tilde{Z}_i$ are independent, and $\tilde{Y}_i$ have common distribution with values in H and that $\tilde{Z}_i$ have common distribution with values in A_e and that $P(\tilde{Z}_1=0)>0$. Next

$$\sup_{g\in G} P((\prod_{i=1}^n X_i=g)\cap A)\leq P(l_{[c_1n]}>n)+\sup_{g\in G} P(\prod_{i=1}^{[c_1n]}(\tilde{Y}_i\tilde{Z}_i)=g)$$
$$\leq \exp(-c_3n)+\exp(-c_2n^{d/(d+2)})$$

where the estimate for the second term follows from (1.2) . Estimate on B is similar, thus giving the upper estimate.

To get the lower bound, we first note, that as in (1.2) , it is enough to estimate

$$P(\prod_{j=1}^n X_j\in H).$$

Morover, we may add to p_1 atom in 0. Indeed, if $q=ap+(1-p)\delta_0$ then

$$q^n=\sum c_{n,k}p^k$$

where $c_{n,k}$ are binomial coefficients. There exists $b>0$ such that for large n

$$\sum_{2k<bn} c_{n,k}r\leq\exp(-bn)$$

Since $p^{2k}(0)$ decreases and $p^{2k+1}(0) = 0$

$$q^n(0) \le \sum_{k \ge bn} c_{n,2k} p^{2k}(0) + \exp(-bn) \le$$

$$p^{2[bn]}(0) + \exp(-bn)$$

so any subexponential lower bound for $q^n(0)$ will give corresponding lower bound for $p^n(0)$.

We put

$$\tilde{p}_1(x) = \frac{2d+2}{2d+3} p_1 + \frac{1}{2d+3} \delta_{(0,0)}$$

Then $P(Z = (0, \pm\delta_0)) = P(Z = (0,0)) = 1/3$.

Next, we restrict our attention to A. We are going to lenghten the product to get fixed number of series, and then to apply (1.2) . Since $\prod_{i=1}^n X_i$ and ω seem to be dependent, we consider conditional distribution and then take expectation in ω. As in the proof of (1.2) , we note that the second (in F) coordinate of $\prod X_i$ has a (conditional in ω and Y_j, $j = 1, \ldots$) distribution which is a cartesian product of one dimensional distributions. Each of those one dimensional distributions is a convolution power of distribution of Z. Since the distribution q of Z is unimodal (thanks to the added atom at 0), its convolution powers have maximum at 0 for any $k \ge 0$. Hence $q^{\star k}(0)$ is decreasing in k. Consequently

$$P((\prod_{i=1}^n X_i \in H) \cap A | \omega, Y_1, Y_2, \ldots) \ge P((\prod_{i=1}^{l_{n+1}-1} X_i \in H) \cap A | \omega, Y_1, Y_2, \ldots).$$

Next, with the notation as in the proof of upper bound

$$P((\prod_{i=1}^n X_i \in H) \cap A) \ge P((\prod_{i=1}^{l_{n+1}-1} X_i \in H) \cap A) = P((\prod_{i=1}^n (\tilde{Y}_i \tilde{Z}_i) \in H) \cap A)$$

and we end the proof applying (1.2) .

To get the last claim of (1.1) we consider $H = G_1$. One easily checks that the number of sites visited by the random walk on G_1 is larger then th corresponding number on Z^d, for any d, so in the final estimate we can take d as large as we wish.

References

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