TRANSFERENCE FOR AMENABLE ACTIONS

WALDEMAR HEBISCH AND M. GABRIELLA KUHN

Abstract. Suppose G acts amenably on a measure space X with quasi-invariant σ-finite measure m. Let σ be an isometric representation of G on $L^p(X, dm)$ and μ a finite Radon measure on G. We show that the operator $\sigma_\mu f(x) = \int_G (\sigma(g)f)(x)d\mu(g)$ has $L^p(X, dm)$-operator norm not exceeding the $L^p(G)$-operator norm of the convolution operator defined by μ. We shall also prove an analogue result for the maximal function M associated to a countable family of Radon measures μ_n.

1. Introduction

Let σ be a uniformly bounded representation of a locally compact group G on some Banach space B and K a convolution operator on $L^p(G)$ given by

$$Kf(x) = \int_G k(y)f(y^{-1}x)dy = k * f(x)$$

with $k \in L^1(G)$. The *transferred operator* $K^\sharp$ is defined by letting

$$K^\sharp(F) = \int_G k(y)(\sigma(y)F)dy \ .$$

The basic transference theory goes back to the earliest works of R. Coifman and G. Weiss around 1970 and says that if we have an action of an *amenable* group G on some (σ-finite) measure space X and we let $B = L^p(X)$ then

$$\|K^\sharp\|_{L^p(X)\to L^p(X)} \le \|K\|_{L^p(G)\to L^p(G)} \ . \tag{1}$$

This theory has many different applications (it allows to convert $L^p(\mathbb{T})$- Fourier multipliers into $L^p(\mathbb{R})$- Fourier multipliers, to bound Riesz transform on $SU(2)$ or, more generally, on compact Lie groups having a "nice" maximal torus) and has been generalized to different contexts: the central role is played by an amenable group.

In 1978 R. Zimmer [8] introduced the notion of *amenable action* for locally compact topological groups G. When X is a homogeneous G space, say $X = G/H$, amenability of the action is equivalent to the amenability of H.

In this paper we shall prove that the above inequality (1) is true when G *acts amenably* on a measure space X with quasi-invariant σ-finite measure. We shall also prove an analogue result for the maximal function M_σ with respect to a countable family of Radon measures μ_n on G.

2. Notation and Results

Let G be a second countable locally compact group acting on a Borel space X with quasi invariant σ-finite measure m. We shall assume that G is acting on the right, that is $(g, x) \in G \times X \to g^{-1}x$. Let $r(g, x)$ be the Radon-Nikodym derivative of the G action:

$$r(g,x) = \frac{dm_g(x)}{dm} \qquad \text{where } m_g(E) = m(g^{-1}E) \tag{2}$$

equivalently

$$\int_X |f(g^{-1}x)| r(g,x) dm(x) = \int_X |f(x)| dm(x)$$

for every measurable function f. Recall that r can be chosen to be a strict Borel cocycle, that is $r(gs, x) = r(g, x)r(s, g^{-1}x)$ for all $g\,, s \in G$ and $x \in X$.

We shall assume that m is a probability measure, so that

$$\int_X r(g,x) dm(x) = 1 \qquad \text{for all } g \in G \ .$$

This can be done by choosing appropriately a representative in the measure class of m.

Let α be a cocycle, that is a Borel function from $G \times X \to \mathbf{C}$ such that:

$$\alpha(gs, x) = \alpha(g, x)\alpha(s, g^{-1}x) \ .$$

In the sequel we will assume that α is unitary, that is $|\alpha(g, x)| = 1$.

For any given $g \in G$ define a norm preserving map $\sigma(g) : L^p(X) \to L^p(X)$ by letting

$$\sigma(g)f(x) = f(g^{-1}x)(r(g,x))^{\frac{1}{p}} \alpha(g,x) \ .$$

Denote by $L^p(G\times X)$ the usual L^p space with respect to the product measure $dg\times dm$ where dg is the left invariant Haar measure on G.

Denote by λ the left regular representation of G acting on $L^p(G)$. Define a representation $\lambda\otimes\sigma$ on $L^p(G\times X)$ by the rule

$$(\lambda\otimes\sigma)(g)F(h,x)=F(g^{-1}h,g^{-1}x)(r(g,x))^{\frac{1}{p}}\alpha(g,x)\ .$$

Choose any finite Radon measure μ on G. Define operators $(\lambda\otimes\sigma)_\mu$, λ_μ and σ_μ acting respectively on $L^p(G\times X)$, $L^p(G)$ and $L^p(X)$ according to the following rules:

$$(\lambda\otimes\sigma)_\mu F(h,x)=\int_G\lambda\otimes\sigma(g)F(h,x)d\mu(g)\int_G F(g^{-1}h,g^{-1}x)(r(g,x))^{\frac{1}{p}}\alpha(g,x)d\mu(g)$$

$$\lambda_\mu f(h)=\int_G\lambda(g)f(h)d\mu(g)=\int_G f(g^{-1}h)d\mu(g)$$

$$\sigma_\mu f(x)=\int_G\sigma(g)f(x)d\mu(g)=\int_G f(g^{-1}x)(r(g,x))^{\frac{1}{p}}\alpha(g,x)d\mu(g)\ .$$

We shall denote by $\|\cdot\|_p$ the L^p norm in any of the above defined L^p spaces and by by $\|\cdot\|_{p,p}$ the norm of any of the above defined operators.

When G is *amenable* it is known (see e.g. [2][3]) that $\|\sigma_\mu\|_{p,p}\le\|\lambda_\mu\|_{p,p}$. This phenomenon is called transference.

When $p=2$ and G *acts amenably on* X the same inequalities are true: see [1] Theorem 3.2.3 (for discrete groups see also [5] for ergodic actions and [6] for general amenable actions including semisimple algebraic groups).

In this paper we shall prove transference for all p $(1\le p<\infty)$.

We refer to Zimmer's book [9] for the general definition of amenable action.

For the purposes of this paper we need the following property of amenable G-spaces:

Lemma 2.1. *[C. Anantharaman-Delaroche]*[1] *Theorem 3.1.5*

Assume that G is acting amenably on a standard Borel space with quasi-invariant measure m.

Then there exists a sequence ϕ_i of nonnegative functions $\phi_i(g,x)$ in $L^1(G\times X)$ such that for all $f\in L^1(X)$ and any compact subset $K\subset G$ we have

a) $\int_G \phi_i(g,x)dg = 1$ *for almost all* $x \in X$

b) $\lim_i(\sup_{k\in K} \int_G \int_X |\phi_i(k^{-1}g, k^{-1}x) - \phi_i(g,x)|\,|f(x)|dm(x)\,dg) = 0$

Remark 2.2. When G is discrete and ergodic on X the existence of such a sequence was essentially the proof (given in [5]) that the quasi-regular representation of G on X is weakly contained into the regular representation of G. In [1] it is proved that the above condition is in fact equivalent to the amenabilty of the G action on X for any locally compact second countable group.

Fix once and for all a sequence ϕ_i of nonnegative functions satisfying (**a**) and (**b**) of Lemma 2.1.

For every $f \in L^p(X)$ and $i \in \mathbb{N}$ define a function $\tilde{f}_i \in L^p(G\times X)$ according to the following rule:

$$\tilde{f}_i(g,x) = (\phi_i(g,x))^{\frac{1}{p}} f(x)$$

so that

$$\widetilde{\sigma_\mu f}_i(g,x) = (\phi_i(g,x))^{\frac{1}{p}}(\sigma_\mu f)(x)\ .$$

Lemma 2.3. *Let μ be any finite Radon measure and $f \in L^p(X)$. Let $\tilde{f}_i$ and $\widetilde{\sigma_\mu f}_i$ be as above.*

Then we have

$$\lim_i \|(\lambda\otimes\sigma)_\mu \tilde{f}_i - \widetilde{\sigma_\mu f}_i\|_p = 0.$$

Proof. First, we will assume that μ has compact support F. Choose $\varepsilon > 0$ and fix f. By Lemma 2.1 (with $K = F^{-1}$) we can find N such that for all $i \geq N$

$$\int_G \phi_i(g,x)dg = 1$$

$$\sup_{k\in F}(\int_G \int_X |\phi_i(kg,kx) - \phi_i(g,x)||f(x)|^p\ dm(x)\ dg) < \varepsilon.$$

We need to compute the norm in $L^p(G\times X)$ of the quantity

$$((\lambda\otimes\sigma)_\mu \tilde{f}_i - \widetilde{\sigma_\mu f}_i)(g,x) =$$
$$\int_F \{(\phi_i(k^{-1}g, k^{-1}x))^{\frac{1}{p}} - (\phi_i(g,x))^{\frac{1}{p}}\}\{r(k,x)\}^{\frac{1}{p}}\alpha(k,x)\ f(k^{-1}x)\,d\mu(k)\ .$$

Denote by $|\mu|$ the measure total variation of μ. Using Minkowski's integral inequality we get

$$\{\int_{G\times X} |(\lambda\otimes\sigma)_\mu \tilde{f}_i - \widetilde{\sigma_\mu f}_i|^p(g,x)\, d(g)\, dm(x)\}^{\frac{1}{p}} \le$$

$$\int_F d|\mu|(k)$$
$$\{\int_{G\times X} |(\phi_i(k^{-1}g,k^{-1}x))^{\frac{1}{p}} - (\phi_i(g,x))^{\frac{1}{p}}|^p r(k,x)|f(k^{-1}x)|^p\, d(g)\, dm(x)\}^{\frac{1}{p}}$$
$$= \int_F \{\int_{G\times X} |\phi_i(kg,kx)^{\frac{1}{p}} - \phi_i(g,x)^{\frac{1}{p}}|^p |f(x)|^p\, d(g) dm(x)\}^{\frac{1}{p}}\, d|\mu|(k)$$
$$\le \int_F \{\int_{G\times X} |\phi_i(kg,kx) - \phi_i(g,x)||f(x)|^p\, d(g) dm(x)\}^{\frac{1}{p}}\, d|\mu|(k) \tag{3}$$
$$\le \varepsilon^{\frac{1}{p}} \|\mu\|$$

where for the last inequality (3) we used $|a^{\frac{1}{p}} - b^{\frac{1}{p}}|^p \le |a-b|$ (valid for positive a and b) and $\|\mu\|$ stays for $|\mu|(G)$.

So, we showed that for fixed compactly supported μ and given f we can choose N such that for all $i \ge N$

$$\|(\lambda\otimes\sigma)_\mu \tilde{f}_i - \widetilde{\sigma_\mu f}_i\|_p \le \varepsilon^{\frac{1}{p}} \|\mu\|$$

which gives our claim for compactly supported μ.

Now, let K_n be an increasing sequence of compact sets such that $G = \cup_n K_n$. Let μ_n denote the restriction of μ to K_n.

$$\|\mu - \mu_n\| = \int_{K_n^c} d|\mu| \to 0$$

becouse $|\mu|$ is finite and countably additive. For fixed n

$$\limsup_i \|(\lambda\otimes\sigma)_\mu \tilde{f}_i - \widetilde{\sigma_\mu f}_i\|_p \le$$
$$\limsup_i \Big(\|(\lambda\otimes\sigma)_{\mu_n} \tilde{f}_i - \widetilde{\sigma_{\mu_n} f}_i\|_p + \|f\|_p \|\mu - \mu_n\|\Big)$$
$$= \|f\|_p \|\mu - \mu_n\|.$$

Since $\|\mu - \mu_n\|$ is arbitrarily small when n is large we get the claim. □

We are now able to prove our result:

Theorem 2.4. *Let G be a second countable locally compact group acting amenably on a standard Borel space with quasi-invariant σ-finite measure m. Fix a finite Radon measure μ on G. Define, as before,*

$$\sigma_\mu f(x) = \int_X f(g^{-1}x)(r(g,x))^{\frac{1}{p}}\alpha(g,x)d\mu(g) \ .$$

Then the operator norm of σ_μ acting on $L^p(X)$ is bounded by the operator norm of λ_μ acting on $L^p(G)$. In other words

$$\|\sigma_\mu\|_{p,p} \leq \|\lambda_\mu\|_{p,p} \ . \tag{4}$$

Proof. Define

$$(UF)(g,x) = F(g,gx)(r(g^{-1},x))^{\frac{1}{p}}\alpha(g^{-1},x) = \sigma(g^{-1})F(g,x)$$

for every $F \in L^p(G \times X)$. Observe that U intertwines $\lambda \otimes \sigma$ to $\lambda \otimes \mathrm{i}$ where

$$(\lambda \otimes \mathrm{i})(g)F(h,x) = F(g^{-1}h,x) \ .$$

Observe that U^{-1} is also well defined:

$$U^{-1}F(g,x) = \sigma(g)F(g,x)$$

and $\|UF\|_p = \|F\|_p = \|U^{-1}F\|_p$. So that $\|(\lambda \otimes \sigma)_\mu\|_{p,p} = \|(\lambda \otimes \mathrm{i})_\mu\|_{p,p}$. Since obviously $\|(\lambda \otimes \mathrm{i})_\mu\|_{p,p} \leq \|\lambda_\mu\|_{p,p}$, it is enough to compare the norm of σ_μ with the norm of $(\lambda \otimes \sigma)_\mu$.

Fix $f \in L^p(X)$. Let $\tilde{f}_i$ and $\widetilde{\sigma_\mu f}_i$ be as in Lemma 2.3. Observe that

$$\|f\|_p = \|\tilde{f}_i\|_p \quad \text{and} \quad \|\sigma_\mu f\|_p = \|\widetilde{\sigma_\mu f}_i\|_p \ .$$

Since

$$\|\widetilde{\sigma_\mu f}_i\|_p \leq \|(\lambda \otimes \sigma)_\mu \tilde{f}_i\|_p + \|(\lambda \otimes \sigma)_\mu \tilde{f}_i - \widetilde{\sigma_\mu f}_i\|_p \leq$$
$$\|(\lambda \otimes \sigma)_\mu\|_{p,p} \, \|\tilde{f}_i\|_p + \|(\lambda \otimes \sigma)_\mu \tilde{f}_i - \widetilde{\sigma_\mu f}_i\|_p =$$
$$\|(\lambda \otimes \sigma)_\mu\|_{p,p}\|f\|_p + \|(\lambda \otimes \sigma)_\mu \tilde{f}_i - \widetilde{\sigma_\mu f}_i\|_p$$

we have also

$$\|\sigma_\mu f\|_p \leq \|(\lambda \otimes \sigma)_\mu\|_{p,p}\|f\|_p + \|(\lambda \otimes \sigma)_\mu \tilde{f}_i - \widetilde{\sigma_\mu f}_i\|_p \ .$$

Taking the limit as $i \to \infty$ we get

$$\|\sigma_\mu f\|_p \leq \|(\lambda \otimes \sigma)_\mu\|_{p,p} \ \|f\|_p$$

which gives

$$\|\sigma_\mu f\|_p \ \leq \ \|(\lambda \otimes \sigma)_\mu\|_{p,p} \ \|f\|_p \ = \ \|(\lambda \otimes \mathrm{i})_\mu\|_{p,p} \ \|f\|_p \ \leq \ \|\lambda_\mu\|_{p,p} \ \|f\|_p$$

.

□

Remark 2.5. As an application take $G = SL(2,R)$ and $X = P^1$, the projective line. Since $X \simeq SL(2,R)/P$ where $P = \begin{pmatrix} a & 0 \\ b & a^{-1} \end{pmatrix}$, X is an amenable G space. The G action is given by fractional linear transformations $g^{-1}x = \frac{ax+b}{cx+d}$ for $g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and the Radon–Nikodym derivative $r(g,x)$ is given by $r(g,x) = \frac{1}{(cx+d)^2}$. For every $f \in L^p(X)$ define

$$R_t f(x) = \frac{f(g^{-1}x)}{(cx+d)^{\frac{2}{p}+it}}$$

or, more generally

$$R_\mu f(x) = \int_G \frac{f(g^{-1}x)}{(cx+d)^{\frac{2}{p}+it}} d\mu(g) \ .$$

Then R_t is an isometry for every real t and $1 \leq p < \infty$; while $\|R_\mu\| \leq \|\lambda_\mu\|$.

More generally we can apply our result to boundaries of symmetric spaces: $X = G\ /MAN$ (so we can transfer to the corresponding principial series representations).

We can also handle symmetric spaces: $X = G/K$ (however, since K is compact a more direct argument works better).

Remark 2.6. Assume that μ (not necessary given by a Radon measure) is a bounded convolver of $L^p(G)$ for some p $(1 \leq p < \infty)$. Suppose also that G is amenable: another phenomenon related to transference says that such a μ is also a bounded convolver of $L^2(G)$ (see [4]). We want to remark that such phenomenon *does not* hold in the case of amenable actions: take the free group F_r on r generators ($r \geq 2$) acting on itself, this action is amenable since the subgroup $\{e\}$ is compact and $X = F_r/e$. It is well known that there exists a bounded convolver of $\ell^p(F_r)$ which is not a convolver of $\ell^q(F_r)$ for any $q \neq p$: see [7]. Moreover, in the same paper it is shown that for every positive M there exists a function f with finite support such that $\|\lambda_f\|_{2,2} > M\|\lambda_f\|_{p,p}$ (λ denotes, as usual, the left regular representation of F_r).

Theorem 2.7. *Let G and X as in Theorem 2.4. Fix any sequence $\{\mu_n\}$ of finite Radon measures on G. Define*

$$M_\sigma f(x) = \sup_n |\sigma_{\mu_n} f|(x) = \sup_n |\int_X f(g^{-1}x)(r(g,x))^{\frac{1}{p}}\alpha(g,x)d\mu_n(g)| \ .$$

Analogously define

$$M_\lambda f(h) = \sup_n |\lambda_{\mu_n} f|(h) = \sup_n |\int_G f(g^{-1}h)d\mu_n(g)| \ .$$

Then the operator norm of M_σ acting on $L^p(X)$ is bounded by the operator norm of the corrsponding M_λ acting on $L^p(G)$.

Sketch of the proof.

Notation as in Theorem 2.4. It is enough to show that, for every positive integer L

$$\|M_{\{\sigma,L\}}f\|_p \leq \|M_{\{\lambda,L\}}\|_{p,p}\|f\|_p$$

where

$$M_{\{\sigma,L\}}f(x) = \sup_{1\leq n\leq L} |\sigma_{\mu_n} f|(x)$$

and $M_{\{\lambda,L\}}$ is defined analogously. Let

$$\widetilde{M_{\{\sigma,L\}}f}_i(g,x) = (\phi_i(g,x))^{\frac{1}{p}} M_{\{\sigma,L\}}f(x) =$$

$$(\phi_i(g,x))^{\frac{1}{p}} \sup_{1\leq n\leq L} |\int_X f(g^{-1}x)(r(g,x))^{\frac{1}{p}}\alpha(g,x)d\mu_n(g)| =$$
$$\sup_{1\leq n\leq L} |(\phi_i(g,x))^{\frac{1}{p}}\int_X f(g^{-1}x)(r(g,x))^{\frac{1}{p}}\alpha(g,x)d\mu_n(g)| =$$
$$\sup_{1\leq n\leq L} |\widetilde{\sigma_{\mu_n} f}_i|(x) \ .$$

Again we have

$$\|M_{\{\sigma,L\}}f\|_p = \|\widetilde{M_{\{\sigma,L\}}f}_i\|_p$$

Remember the defintion of $U^{-1} : L^p(G\times X) \to L^p(G\times X)$:

$$(U^{-1})F(g,x) = F(g,g^{-1}x)(r(g,x))^{\frac{1}{p}}\alpha(g,x) = \sigma(g)F(g,x)\ .$$

Put

$$VF(g,x) = F(g,gx)(r(g^{-1},x))^{\frac{1}{p}}$$

so that

$$(V^{-1})F(g,x) = F(g,g^{-1}x)(r(g,x))^{\frac{1}{p}}$$

and V is an isometry.

Observe that

$$V^{-1}(\sup_n |F_n|) = \sup_n(|U^{-1}F_n|)\ .$$

For every $f \in L^p(G\times X)$ write $f = U^{-1}F$ and evaluate

$$M_{\{\lambda\otimes\sigma,L\}}f(g,x) = \sup_{1\le L\le n}|(\lambda\otimes\sigma)_{\mu_n}f|(g,x) = \sup_{1\le L\le n}|(\lambda\otimes\sigma)_{\mu_n}U^{-1}F|(g,x) =$$
$$\sup_{1\le L\le n}|U^{-1}(\lambda\otimes \mathrm{i})_{\mu_n}F|(g,x) = V^{-1}\sup_{1\le L\le n}|(\lambda\otimes \mathrm{i})_{\mu_n}F|(g,x) =$$
$$V^{-1}(M_{\{\lambda\otimes \mathrm{i},L\}})F(g,x)\ .$$

Hence

$$\|M_{\{\lambda\otimes\sigma,L\}}\|_{p,p} = \|M_{\{\lambda\otimes \mathrm{i},L\}}\|_{p,p} \le \|M_{\{\lambda,L\}}\|_{p,p}\ . \tag{5}$$

Compute

$$\|M_{\{\sigma,L\}}f\|_p = \|\widetilde{M_{\{\sigma,L\}}f}_i\|_p \le \tag{6}$$
$$\|\widetilde{M_{\{\sigma,L\}}f}_i - M_{\{\lambda\otimes\sigma,L\}}\tilde{f}_i\|_p + \|M_{\{\lambda\otimes\sigma,L\}}\|_{p,p}\|f\|_p\ .$$

Since

(7)

$$\left|\sup_{1\le n\le L}|\widetilde{\sigma_{\mu_n}f}_i|(g,x) - \sup_{1\le n\le L}|(\lambda\otimes\sigma)_{\mu_n}\tilde{f}_i|(g,x)\right| \le \sum_{n=1}^{L}|\widetilde{\sigma_{\mu_n}f}_i - (\lambda\otimes\sigma)_{\mu_n}\tilde{f}_i|(g,x)$$

one has

$$\|\widetilde{M_{\{\sigma,L\}}f}_i - M_{\{\lambda\otimes\sigma,L\}}\tilde{f}_i\|_p \leq \sum_{n=1}^{L} \|\widetilde{\sigma_{\mu_n}f}_i - (\lambda\otimes\sigma)_{\mu_n}\tilde{f}_i\|_p \to 0.$$

Conclude the proof as in Theorem 2.4. □

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Mathematical Institute, Wrocław Uniwersity, pl. Grunwaldzki 2/4, 50-384 Wrocław, Poland

E-mail address: hebisch@math.uni.wroc.pl

Dipartimento di Matematica, Università di Milano "Bicocca", Viale Sarca 202, 20126 Milano, ITALIA

E-mail address: kuhn@matapp.unimib.it