## Linear algebra 1R, problem sheet 10

1. Prove: if two of vectors $X_1, \ldots, X_l$ are equal, then $X_1, \ldots, X_l$ are linearly dependent.
2. Check in different ways if the following sets of vectors are linearly independent: (a) $(1,1,0)^\top$, $(0,1,1)^\top$, $(1,0,1)^\top$; (b) $(1,0,1)^\top$, $(0,1,0)^\top$; (c) $(1,2,-3)^\top$, $(-1,-1,2)^\top$, $(3,-2,-1)^\top$.
3. Check via computation $\det(M) = \det(M^\top)$ (for $3\times 3$ matrix).
4. Compute determinants:
$$\begin{vmatrix} 1 & 2 & 3 \\ 5 & 1 & 4 \\ 3 & 2 & 5 \end{vmatrix}, \begin{vmatrix} -1 & 5 & 4 \\ 3 & -2 & 0 \\ -1 & 3 & 6 \end{vmatrix}, \begin{vmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{vmatrix}, \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}, \begin{vmatrix} a & d & e \\ 0 & b & f \\ 0 & 0 & c \end{vmatrix}, \begin{vmatrix} 1 & 1 & -1 \\ 3 & 5 & 5 \\ -2 & 2 & 2 \end{vmatrix}, \begin{vmatrix} a & -1 & 0 \\ b & x & -1 \\ c & 0 & x \end{vmatrix}.$$

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5. Let $k < l$. Prove: (a) if $X_1, \ldots, X_l$ are linearly independent, then $X_1, \ldots, X_k$ are linearly independent; (b) if $X_1, \ldots, X_k$ are linearly dependent, then $X_1, \ldots, X_l$ are linearly dependent.
6. Prove that, 4 arbitrary vectors in $\mathbf{R}^3$ are linearly dependent.
7. Which of conditions below are equivalent to linear independence of vectors $X_1, X_2, X_3 \in \mathbf{R}^3$? Prove or find counterexample.
   a) No of two vectors among $X_1, X_2, X_3$ are colinear.
   b) $X_1$ is not a linear combination of $X_2, X_3$; $X_2$ is not a linear combination of; additionally $X_3 \neq 0$.
8. Prove: if $X \perp Y \perp Z \perp X$ and $X, Y, Z \neq 0$, then $X, Y, Z$ are linearly independent.
9. Write down first of given vectors as a linear combination of the other (or check that this is impossible):
(a) $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$; $\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}$; (b) $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$; $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$, $\begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$; (c) $\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$; $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, $\begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix}$, $\begin{pmatrix} 1 \\ -2 \\ -5 \end{pmatrix}$;
(d) $\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$; $\begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$, $\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$; (e) $\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$; $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix}$;
10. It is known that $U$: (1) is contained in plane $x-y+2z+3=0$, (2) is orthogonal to vector $W = (1,0,-3)^\top$, (3) parallelepiped determined by vectors $U$, $W$, $(-2,-1,1)^\top$ have volume 72. Find $U$.
11. Compute distance of $(1,2,3)^\top$ from (a) line $\frac{x-1}{3} = \frac{2y+4}{3} = \frac{z+1}{1}$; (b) plane $2x - y + 2 = 5$.
12. Solve systems of equations below using method that you prefer and also using Cramer formula. $(a) \begin{cases} 2x+y+z=3 \\ x+2y+z=0 \\ x+y+2z=9 \end{cases}$
13. Find polynomial $f(x)$ of degree 2, such that $f(1) = 8$, $f(-1) = 2$, $f(2) = 14$.
14. Check that $\det(A+B, B+C, C+A) = 2\det(A,B,C)$, $(x-y)(y-z)(z-x) = \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$.
15. Find out if given sets of vectors are positively oriented or negatively oriented:
(a) $(1,6,3)^\top$, $(-1,0,2)^\top$, $(2,1,1)^\top$ (b) $(1,6,3)^\top$, $(2,-1,1)^\top$, $(-1,7,2)^\top$
16. Prove: if $A$, $B$, $C$ (in given order) are positively oriented, then also $-A$, $-B$, $C$; $A+B$, $B$, $C$ are positively oriented.