Linear algebra 1R, problem sheet 11

1. Prove: $\ker(F\circ G)\supseteq\ker(G)$, $\mathrm{Im}(F\circ G)\subseteq\mathrm{Im}(F)$ where F,G are linear transformations.
2. Find nonparametric equations describing images and kernels of transformations $\mathbf{R}^n\to\mathbf{R}^k$ given by matrices:
$\begin{pmatrix}1&-2&3\\-3&6&-9\end{pmatrix}$; $\begin{pmatrix}5&-1&2\end{pmatrix}$; $\begin{pmatrix}5\\-1\\2\end{pmatrix}$; $\begin{pmatrix}1&0&3\\3&-1&0\\1&1&2\end{pmatrix}$;
3. Find general form of solution of $AX=Y$ (a) for: $A=\begin{pmatrix}1&2&-1\\2&-1&1\\1&7&-4\end{pmatrix}$, $\begin{pmatrix}1&3&-2\\3&9&-6\\-2&-6&4\end{pmatrix}$;
$Y=(0,0,0)^\top,(1,1,1)^\top,(1,-2,3)^\top,(-6,-3,3)^\top,(2,4,2)^\top,$
$(-1,3,0)^\top,(1,0,3)^\top,(1,3,0)^\top,(4,12,-8)^\top$; (b) for $A=\begin{pmatrix}1&-2&3\\-3&6&-9\end{pmatrix}$; $Y=\binom{0}{0},\binom{2}{0},\binom{-5}{15},\binom{90}{-30}$.
4. Prove: if $AX=Y$ has two solutions, then it has infinitely many solutions.
5. Find M^{-1}. Try all methods known to you: $M=$
$\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}$; $\begin{pmatrix}1&0&0\\0&0&-1\\0&1&0\end{pmatrix}$; $\begin{pmatrix}5&5&1\\1&0&0\\0&0&1\end{pmatrix}$; $\begin{pmatrix}1&2&3\\1&0&-1\\0&1&1\end{pmatrix}$; $\begin{pmatrix}1&-1&2\\6&0&1\\3&2&1\end{pmatrix}$; $\begin{pmatrix}1&a&b\\0&1&c\\0&0&1\end{pmatrix}$; $\begin{pmatrix}a&0&0\\0&b&0\\0&0&c\end{pmatrix}$.

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6. Find matrix of (a) (orthogonal) projection onto plane $3x+2y-z=0$; (b) (orthogonal) projection onto line $X=t(1,0,-7)^\top$; (c) symmetry with respect to line $X=t(-1,2,1)^\top$.
7. Write down matrices of rotation by angle θ around x axis and around y axis. For $\theta=\pi/2$ compute superposition of those rotations in two possible orders.
8. Find matrix of rotation by angle $2\pi/3$ around line $x=y=z$. (There are two such rotations (depending on orientation); choose one.)
9. Let $F:\mathbf{R}^2\to\mathbf{R}^3$ and $G:\mathbf{R}^3\to\mathbf{R}^2$ be linear transformations
   a) Prove that F is not onto (surjective).
   b) Prove that G is not injective.
   c) Prove that $F\circ G$ is not invertible.
10. Give examples of linear transformations $F,G:\mathbf{R}^3\to\mathbf{R}^3$, such that (a) $\ker(F)\subset\mathrm{Im}(F)$ (b) $\mathrm{Im}(G)\subset\ker(G)$. Try to find both trivial and nontrivial examples.
11. Give example of linear transformation $F\colon\mathbf{R}^3\to\mathbf{R}^3$, which has a plane as an image, kernel is a line and angle between kernel and image is $60°$. Write down matrix of this transformation.
12. It is known that $F,G,H:\mathbf{R}^3\to\mathbf{R}^3$ are linear transformations such that $F\circ G\circ H$ is surjective, $\det(G)=-4$, $H=F\circ G\circ F$. Prove that F is injective.
13. Prove: if $N^3=0$, then $(I+N)^{-1}=I-N+N^2$. Use this formula to compute $\begin{pmatrix}1&2&5\\0&1&7\\0&0&1\end{pmatrix}^{-1}$.
14. Are the following formulas true (for invertible 3×3 matrices): $(MN)^{-1}=M^{-1}N^{-1}$, $(M+N)^{-1}=M^{-1}+N^{-1}$, $\det(aM)=a\det(M)$, $\det(M+N)=\det(M)+\det(N)$?
15. Prove that for every vector $A\in\mathbf{R}^3$ transformation M_A given by formula $M_A(X)=A\times X$ is linear. Write down its matrix in terms of coordinates of A.

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16. Compute determinant of matrix $\begin{pmatrix}121&-248\\-321&625\\144&-91\end{pmatrix}\begin{pmatrix}321&231&123\\-619&26&-17\end{pmatrix}$. Hint: answer is immediate, without computation.
17. Let $M,N\in M_{3\times3}$. It is known that $\det(MN)\neq0$. Prove that M i N are invertible.
18. Let $F:\mathbf{R}^3\to\mathbf{R}^3$ be a linear transformation. Prove: if $\ker(F)$ is one point/a line/a plane/$\mathbf{R}^3$, then $\mathrm{Im}(F)$ is $\mathbf{R}^3$/a plane/a line/one point, respectively.