Linear algebra 1R, problem sheet 12

All matrices are of dimension 3×3, unless dimension is specified otherwise.

1. Prove the following formulas. Assume that needed inverses exist. a) $(M^{-1})^{-1} = M$; b) $\det(MNM^{-1}) = \det(N)$; c) $(\lambda I - M)^{-1} - (\mu I - M)^{-1} = (\mu - \lambda)(\mu I - M)^{-1}(\lambda I - M)^{-1}$; d) $\det(M^{-1}) = (\det(M))^{-1}$.
2. Compute eigenvalues and eigenvectors. Compare multiplicity of root of characteristic polynomial with maximal possible number of corresponding linearly independent eigenvectors.
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}; \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}; \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix}; \begin{pmatrix} 0 & 1 & 0 \\ -4 & 4 & 0 \\ -2 & 1 & 2 \end{pmatrix}.$$
3. Find eigenvalues and eigenvectors of (a) rotation by angle θ around some line; (b) orthogonal projection onto a line; (c) orthogonal projection onto a plane; (d) symmetry (reflection) with respect to a line; (e) symmetry (reflection) with respect to a plane. All lines and planes above pass trough 0.
4. Vectors $(1, -2, 2)^\top, (-3, 0, 1)^\top, (7, 1, 0)^\top$ are eigenvectors of linear transformation $T : \mathbf{R}^3 \to \mathbf{R}^3$ with eigenvalues $-1, -3, 2$ respectively. Find matrix M of T.
5. Find all rational roots of polynomials below. Find out how many real root they have. (a) x^3+5x^2+2x-8, (b) $x^3 + 5x^2 + 5x + 4$, (c) $2x^3 - 10x^2 + 7x + 3$, (d) $6x^3 + 7x^2 - x - 2$, (e) $-x^3 - 7x^2 + 3x + 27$, (f) $-x^3 + 7$, (g) $2x^3 - (1 + 2\sqrt{2})x^2 + (\sqrt{2} - 1)x + \sqrt{2}$.
6. Write down at least 3 matrices of linear isometry which have $(3/5, \sqrt{7}/5, -3/5)$ as the first row.

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7. Matrices A i A^{-1} have integer entries. What can you say about $\det(A)$?
8. Let $X, Y, Z \in \mathbf{R}^3$ be nonzero vectors, A be a matrix, $\lambda, \mu \in \mathbf{R}$ be two not equal numbers. It is known that $AX = \lambda X$, $AY = \lambda Y$, $AZ = \mu Z$. Prove: if X, Y are linearly independent, then X, Y, Z are linearly independent.
9. Diagonalize matrix M: find eigenvalues and eigenvectors; write down M in form PDP^{-1} with diagonal D; find new coordinates in which M is diagonal; express new coordinate in terms of old and old in terms of new. $M = \begin{pmatrix} 1 & 4 & -1 \\ 0 & 3 & -5 \\ 0 & 0 & -2 \end{pmatrix}$; $M = \begin{pmatrix} 2 & -1 & 2 \\ 5 & -3 & 3 \\ -1 & 0 & -2 \end{pmatrix}$.
10. Which of following matrices can be diagonalized: $\begin{pmatrix} 4 & -5 & 2 \\ 5 & -7 & 3 \\ 6 & -9 & 4 \end{pmatrix}$, $\begin{pmatrix} 7 & -12 & 6 \\ 10 & -19 & 10 \\ 12 & -24 & 13 \end{pmatrix}$.
11. Sequence (a_n) is given recursively: $a_0 = 1$, $a_1 = 7$, $a_2 = -1$, $a_{n+3} = 4a_{n+2} - a_{n+1} - 6a_n$. Derive explicit formula for a_n.
12. The following transformations are isometries of $\mathbf{R}^3$; write down each as a superposition of translation and linear isometry. Write down matrix of linear part. (a) reflection with respect to plane $x + z = 1$; (b) reflection with respect to line $x = z = 1$; (c) rotation by angle $\pi/2$ around line $y = z = 1$ (choose one of two possible rotations).
13. Find matrix of linear isometry F, such that $F((2, 2, -1)^\top) = (0, 3, 0)^\top$.
14. Find matrix of linear isometry F, such that $F((2, 2, -1)^\top) = (1, -1, \sqrt{7})^\top$. How many answers are there?
15. Trace of a linear isometry is -3. Find the isometry.
16. Find matrix of rotation by angle $\frac{\pi}{6}$ around line $x = -y = \frac{z}{3}$ (choose direction of rotation). Then explain in which direction you are rotating.
17. Let A be linear isometry of $\mathbf{R}^3$ which changes orientation and such that 1 is its eigenvalue. Prove that A is a reflection with respect to some plane.

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18. Assume that matrix A has 3 different eigenvalues, and $AB = BA$. Prove that matrix B can be written in diagonal form.