Linear algebra 1R, problem sheet 13

1. Check that given matrix matrix of an isometry. Is this isometry a rotation or is it a composition of reflection with respect to a plane and a rotation? Find axis and angle.

$$\begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix}; \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}.$$

2. Write down a quadratic form Q on $\mathbf{R}^3$ such that $Q((1,2,3)^\top) = -7$.
3. Let $Q((x,y,z)^\top) = xy + yz + zx$. Find vector U such that: (a) $Q(U) = 1$; (b) $Q(U) = -1$; (c) $Q(U) = 0$.
4. Draw surfaces: $x^2 = 0$, $x^2 + 4y^2 + 4z^2 = 0$, $x^2 + z^2 = 0$, $x^2 + z^2 = 1$, $x^2 - y^2 = 0$, $x^2 + y^2 - z^2 = -1$, $x^2 + y^2 + z^2 = 0$, $-x^2 + y^2 + z^2 = 1$, $z - y^2 = 0$.
5. Give an example of equation of degree 2 in three variables which has a line (in $\mathbf{R}^3$) as a solution set.

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6. Find symmetric matrix $A \in M_{3\times3}(\mathbf{R})$ such that $A\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$, $A\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}$ and number 3 is an eigenvalue of A.
7. Find symmetric matrix $A \in M_{3\times3}(\mathbf{R})$ such that $\chi_A(x) = 2x^2 - x^3$ and each vector in the plane $2x + y + z = 0$ is an eigenvector of A.
8. Give an example of symmetric matrix S of dimension 3×3 and its three linearly independent eigenvectors such that no two of them are orthogonal.
9. Assume that X is an eigenvector of matrix $A \in M_{3\times3}(\mathbf{R})$ such that $A^\top = I - A$. Prove: if $Y \perp X$, then $AY \perp X$.
10. Assume that $A \in M_{3\times3}(\mathbf{R})$ is symmetric, $AX = \lambda X$, $AY = \mu Y$, $\langle X, Y \rangle \neq 0$. Prove that $\lambda = \mu$. Infer from this that each linear combination of vectors X and Y is an eigenvector of A.
11. Prove: if linearly independent vectors $X, Y \in \mathbf{R}^3$ are eigenvectors of a symmetric matrix $A \in M_{3\times3}(\mathbf{R})$, then vector $X \times Y$ is an eigenvector of A.
12. Give an example of quadratic form Q on $\mathbf{R}^3$ such that $Q(E_1), Q(E_2), Q(E_3) > 0$, but for some $U \in \mathbf{R}^3$ we have $Q(U) < 0$.
13. Let Q be a quadratic form on $\mathbf{R}^3$, and $U, V \in \mathbf{R}^3$ be such that $Q(U) = -Q(V) = 1$. Prove that there exist infinitely many vectors $W \in \mathbf{R}^3$, such that $Q(W) = 0$.
14. It is known that surface of degree 2 given by equation $ax^2 + 2bxy + 2cxz + dy^2 + 2eyz + fz^2 + g = 0$ may be an ellipsoid, a one sheet hyperboloid, a two sheet hyperboloid, ... Complete the list considering all possible canonical forms of this equation. Include also degenerate cases like a point.