Linear algebra 1R, problem sheet 14

1. Justify: if matrix $A \in M_{3\times 3}(\mathbf{R})$ has a nonreal eigenvalue, then A can be put in diagonal form over $\mathbf{C}$.
2. Give an example of hyperbolic paraboloid and a line on it.

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3. Transform surfaces below to canonical form: write down matrix of corresponding quadratic form; find eigenvalues and orthogonal basis consisting of eigenvectors; write down canonical form of the equation and transformation matrices linking coordinates x, y, z and x', y', z' where the equation has canonical form. Identify the surfaces, draw them in x', y', z'-coordinates. Using canonical form find out geometric properties of the surface (length of half-axes, symmetries or lack of them, distance between sheets, angle of asymptotic cone etc.). Describe position of the surfaces in x, y, z-coordinates (giving direction of axes, axis of symmetry or of asymptotic cone etc.).
(a) $x^2 - 5y^2 + z^2 + 4xy + 2xz + 4yz = 3$,
(e) $11x^2 + 4xy - 16xz + 2y^2 + 20yz + 5z^2 = 9$,
(f) $-2xz + y^2 = 1$,
(g) $5x^2 - 2xy - 2xz + 5y^2 - 2yz + 5z^2 = 3$,
4. Like above but for:
(c) $x^2 - 4y^2 - 4yz - z^2 = 0$,
(d) $6x^2 + 5y^2 + 7z^2 - 4xy + 4xz = 0$,
(h) $x^2 + y^2 + 5z^2 - 6xy - 2xz + 2zy = 0$,
(j) $2x^2 - 8xy + 8y^2 + 12xz - 24yz + 18z^2 = 1$.
5. For each polynomial given below find number of positive roots and number of negative roots (and check if 0 is a root). Find multiplicity of roots. (Hint: compute derivative; find its extremal values, intervals of monotonicity, values of the polynomial at endpoints and value at 0. Frequently it is enough to compute values of polynomial at few points.) (a) $-x^3 + 7x^2 + 2x - 1$, (b) $x^3 - 4x^2 + 17x - 1$, (c) $x^3 + 3x$, (d) $x^3 + x^2 - 8x - 12$, (e) $x^3 + 2x^2 - x + 2$, (f) $x^3 + x^2 + x + 3$.
6. Identify surfaces (for example as an ellipsoid, a pair of planes, etc.) without computing full canonical form, but only by finding out needed signs.
(a) $x^2 - 2xy + 2y^2 + 4xz + 3z^2 = 1$,
(b) $x^2 - 2xy + 2xz + 4yz + 3z^2 = 0$,
(c) $x^2 + 2xy + 2y^2 + 2xz + 2yz + 4z^2 = 1$,
(d) $-2x^2 + 2xy + y^2 + 2xz + 6yx - 2z^2 = 1$.
7. Assume that matrices $A, B \in M_{3\times 3}(\mathbf{R})$ both have characteristic polynomial $(3-x)(5-x)^2$, but neither one can be put in diagonal form. Prove that there exists invertible matrix P such that $A = PBP^{-1}$.
8. Transform to Jordan form:
$$\begin{pmatrix} 1 & -3 & 4 \\ 4 & -7 & 8 \\ 6 & -7 & 7 \end{pmatrix}; \quad \begin{pmatrix} 4 & -5 & 7 \\ 1 & -4 & 9 \\ -4 & 0 & 5 \end{pmatrix}; \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -3 & 3 \end{pmatrix}.$$

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9. Let Q be a quadratic form on $\mathbf{R}^2$ and $F : \mathbf{R}^3 \to \mathbf{R}^2$ a linear transformation. (a) Check that $Q \circ F$ is a quadratic form form $\mathbf{R}^3$. (b) Show that 0 is one of coefficients of canonical form of $Q \circ F$.
10. For few surfaces below find canonical form; name the surface; find rectangular (not necessarily linear) coordinate system X' such that equation has canonical form, that is find matrix of isometry P and vector T such that $X' = P^\top X + T$; draw surfaces in this coordinate system; find position of the surface in standard coordinate system using sentences like "the symmetry axis of hyperboloid is line $X = (1,2,3)^\top + t(3,4,5)^\top$".
(a) $x^2 + 2xy + y^2 - z^2 + 2z - 1 = 0$,
(b) $4x^2 - y^2 - z^2 + 32x - 12z + 44 = 0$,
(c) $z^2 = 2xy$,
(d) $2xy + 2yz + 2zx + 2x + 2y + 2z + 1 = 0$,
(e) $3x^2 + 3y^2 + 3z^2 - 2xy - 2yz - 2zx - 2x - 2y - 2z - 1 = 0$,
(f) $4x^2 + y^2 + 4z^2 - 4xy - 8xz + 4yz - 28x + 2y + 16z + 45 = 0$,
(g) $4x^2 + 4y^2 - 8z^2 - 10xy + 4xz + 4yz - 16x - 16y + 10z - 2 = 0$.
11. Give an example that intersection of the cone $x^2 + y^2 - z^2 = 0$ and a plane may be an ellipsis, a hyperbole, a parable.