## Linear algebra 1R, Problem sheet 2

1. Express $\binom{4}{-2}$ as a linear combination of $\binom{1}{1}$ and $\binom{-1}{1}$.
2. Let $A = \binom{1}{0}$, $B = \binom{1}{1}$, $C = \binom{-1}{0}$. If possible, express (a) $A$ as a linear combination of $B$ i $C$, (b) $B$ as a linear combination of $A$ i $C$, (c) $C$ as a linear combination of $B$ i $A$.
3. Find all $x$ such that the following pair of vectors is linearly dependent.
(a) $\binom{x}{1}, \binom{9}{3}$, (b) $\binom{x}{x^2}, \binom{-2}{4}$, (c) $\binom{x}{1}, \binom{4}{x}$, (d) $\binom{x}{x^2}, \binom{1}{x}$.
4. Show that if $U$ and $V$ linearly independent, then so are $U$ and $U+V$.
5. Use Cramer's formulae to solve $\begin{cases} -2x+3y=4 \\ 4x+6y=-2 \end{cases}$.

---

6. Suppose that $\det\left(\binom{1}{0}, U\right) = 1$, and that $\|U\| = 2$. What can you say about $U$?
7. Check that the determinant of a pair of vectors is bilinear and antisymmetric. Check the validity of Cramer's formulae.
8. Is the pair $U = \binom{-3}{2}$, $V = \binom{-4}{3}$ positively oriented? Find the area of the parallelogram spanned by $U$ and $V$, as well as the area of the triangle $OUV$.
9. Find the area of the triangle with vertices $\binom{-5}{2}$, $\binom{1}{4}$, $\binom{3}{-6}$.
10. For what values of $a$, $b$ the system $\begin{cases} -2x+3y=a \\ 4x-6y=b \end{cases}$ is solvable?
11. Let $A = \binom{2}{5}$ and $B = \binom{5}{1}$ be two vertices of a parallelogram of area $s = 17$. Find the coordinates of the other two vertices, assuming that the intersection point of the diagonals of the parallelogram is on the $OY$ axis.
12. Let $A = \binom{-2}{-3}$, $B = \binom{3}{2}$, $M = \binom{2}{0}$, $N = \binom{-4}{1}$. Consider the signs of suitable determinants to find out whether the points $M$ and $N$ lie on the same side of the line $AB$.
13. Do there exists vectors $U, W$ in the plane such that $\langle U, W\rangle = 3$, $\|U\| = 4$, $\|W\| = 5$? What if we change 3,4,5 to 3,2,1? Generalize.
14. Discuss the existence and the number of solutions of a system of 2 linear equations with two unknowns with determinant $W = 0$. Express your claims in various terms (like: lines crossing, linear independence of vectors, vanishing of determinants $W_x$, $W_y$, $W$). Give examples.
15. Show that a $2\times 2$ system of linear equations with zero right hand sides has a non-zero solution if and only if its determinant $W = 0$.
16. What geometric figures can be obtained as sets of all linear combinations of two given vectors in $\mathbf{R}^2$?

---

17. Suppose that a $2\times 2$ linear system with rational coefficients and rational right-hand sides has a solution different from $(0,0)$. Show that this system has a solution $(x,y) \neq (0,0)$ with rational $x$ and $y$.
18. Let $F : \mathbf{R}^2 \times \mathbf{R}^2 \to \mathbf{R}$ be a function assigning real numbers to pairs of vectors. Suppose that $F$ is bilinear, and that for each $U \in \mathbf{R}^2$ we have $F(U,U) = 0$. Prove that $F$ is anti-symmetric.
19. Let $F : \mathbf{R}^2 \times \mathbf{R}^2 \to \mathbf{R}$ be a function assigning real numbers to pairs of vectors. Suppose that $F$ is bilinear and anti-symmetric. Prove that there exists a constant $C \in \mathbf{R}$, such that for all $U, V \in \mathbf{R}$ we have $F(U,V) = C \cdot \det(U,V)$.
20. Three numbers are written on the board. We are allowed to replace one of them by its sum with a rational multiple of the difference of the other two numbers. Is it possible to pass, by a sequence of such operations, from the triple $0, 1, \sqrt{2}$ to the triple $0, \sqrt{2}, 2$? (Triples are unordered.)