Linear algebra 1R, Problem sheet 3

1. a) Check that the map $\binom{x}{y} \to \binom{ax+by}{cx+dy}$ is additive and homogeneous.
   b) Check that for any matrices M, N, L and any vector U: $(MN)U = M(NU)$, $L(MN) = (LM)N$.
   c) Check by direct calculation: $\det(MN) = \det M \det N$.
2. Compute MN and NM for (a) $M = \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix}$, $N = \begin{pmatrix} 3 & -4 \\ 2 & 1 \end{pmatrix}$, (b) $M = \begin{pmatrix} 1 & 2 \\ 7 & 1 \end{pmatrix}$, $N = \begin{pmatrix} 1 & 1 \\ 0 & 3 \end{pmatrix}$.
3. Find matrices of the maps P_U and S_U for $U = \binom{-1}{3}$.
4. Draw in the coordinate system the image of the lattice $\{\binom{x}{y} : (x \in \mathbf{Z}) \vee (y \in \mathbf{Z})\}$ under the linear map given by the following matrix: (a) $\begin{pmatrix} 4 & -1 \\ 1 & 2 \end{pmatrix}$; (b) $\begin{pmatrix} -2 & 1 \\ -3 & 3 \end{pmatrix}$.
5. Compute M^{-1} for the following M: $\begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix}$, $\begin{pmatrix} 3 & -4 \\ 2 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 2 \\ 7 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 1 \\ 0 & 3 \end{pmatrix}$, $\begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix}$.

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6. Express the following map by a formula in coordinates:
   (a) $P_{\binom{1}{0}}$, (b) $T_{\binom{3}{4}} \circ P_{\binom{1}{0}}$, (c) $R_\pi \circ T_{\binom{1}{2}}$, (d) $R_{\pi/3}$.
7. Find the matrix of the linear map F given that (a) $F\binom{5}{0} = \binom{3}{1}$, $F\binom{0}{7} = \binom{-2}{3}$, (b) $F\binom{4}{1} = \binom{2}{3}$, $F\binom{1}{-1} = \binom{0}{1}$.
8. Prove or disprove: $\det(A + B) = \det(A) + \det(B)$.
9. Find a linear map $J\colon \mathbf{R}^2 \to \mathbf{R}^2$, such that for all $U, V \in \mathbf{R}^2$ we have $\det(U, V) = \langle U, J(V)\rangle$.
10. Find all matrices M such that $M \begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix} M$.
11. Show that multiplying an arbitrary matrix M on the left by $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ exchanges the rows of M; multiplying M by the same matrix on the right—exchanges the columns. Describe (in words) what happens to M after multiplying by $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ on the right.
12. Solve the given matrix equation using inverse matrices:
$$(a)\ \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix} M = \begin{pmatrix} 4 & -6 \\ 2 & 1 \end{pmatrix}, \quad (b)\ \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} M \begin{pmatrix} 1 & 3 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 3 \\ 2 & 2 \end{pmatrix}.$$
13. Let S_x, S_y be reflections in the coordinate axes; P_x, P_y—perpendicular projections on the axes; J_r—dilation with scale r (with centre at O); R_θ—rotation by θ (around O).
   (a) Write down the matrices of the above transformations.
   (b) Using matrices show: $R_{\pi/2} \circ S_x \circ R_{-\pi/2} = S_y$, $J_r \circ J_s = J_{rs}$, $R_\theta \circ R_\phi = R_{\theta+\phi}$.
   (c) Use matrices to recognize the following composite maps: $R_{\pi/2} \circ P_y \circ R_{-\pi/2}$, $S_x \circ S_y$, $S_x \circ P_x$, $P_x \circ S_x$, $P_x \circ P_y$.
14. Let $U, W \in \mathbf{R}^2$ be linearly independent
   a) Show that every $X \in \mathbf{R}^2$ can be uniquely expressed as a linear combination of U, W.
   b) Show that if F, G are linear transformations of the plane such that $F(U) = G(U)$ and $F(W) = G(W)$, then $F = G$.
   c) Show that for arbitrary two vectors A, B there exists a (unique) linear transformation of the plane F, such that $F(U) = A$, $F(W) = B$.
15. Prove the following formulae for invertible matrices: $\det(M^{-1}) = \frac{1}{\det(M)}$, $(MN)^{-1} = N^{-1}M^{-1}$.

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16. Describe S_ℓ by an explicit formula in coordinates, where ℓ is the line given by the equation $2x + 3y = 5$.
17. Let A, B be 2×2 matrices. Prove that if $AB = I$, then $BA = I$.
18. Prove that for every invertible matrix A there exists an $\epsilon > 0$, such that if the entries of a matrix B differ from the corresponding entries of A by less than ϵ, then B is also invertible.
19.** Let $F\colon \mathbf{R}^2 \to \mathbf{R}^2$ be a bijection. Let $F(0) = 0$. Assume that the image of every line by F is again a line. Prove that F is linear.