Linear algebra 1R, problem sheet 4

1. Prove: $M(N+P) = MN+MP$, $(M+N)P = MP+NP$, $t(MN) = (tM)N = M(tN)$, $\det(M^{-1}NM) = \det(N)$, $\det(tM) = t^2\det(M)$, $\mathrm{tr}(MN) = \mathrm{tr}(NM)$, $\mathrm{tr}(M+N) = \mathrm{tr}(M) + \mathrm{tr}(N)$, $\mathrm{tr}(M^{-1}NM) = \mathrm{tr}(N)$ (for matrices M, N, P and for $t \in \mathbf{R}$).
2. Show that the set of linear transformations of the palne, equipped with the operations of addition and multiplication by scalars satisfies the vector space axioms.
3. Determine the values of the parameter c for which the matrix $\begin{pmatrix} 2 & c \\ -c & 1 \end{pmatrix}$ has (a) zero, (b) one, (c) two eigenvalues.
4. Find characteristic polynomials, eigenvalues and eigenvectors for the following matrices: (a) $\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$, (b) $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$, (c) $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$, (d) $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, (e) $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, (f) $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$.

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5. Let $A = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. Compute A^2, A^3 i A^4. State the resulting conjecture; prove it by induction.
6. Show that 5 is the unique eigenvalue of $\begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}$ jest 5. Do the same for $\begin{pmatrix} 5 & 1 \\ 0 & 5 \end{pmatrix}$. Show that one of these matrices is diagonalizable, and the other is not.
7. Express $M = \begin{pmatrix} 3 & 0 \\ 4 & 2 \end{pmatrix}$ as PDP^{-1}, where D is diagonal. Use it to compute the sixth power of M.
8. Find the matrix of the linear transformation with eigenvectors $\binom{1}{0}$ (corresponding to the eigenvalue $-\frac{1}{2}$), $\binom{1}{1}$ (for eigenvalue 1).
9. Give an example of a transformation of the plane that preserves area, changes orientation and has eigenvalue π.
10. Which of the following transformations are diagonalizable? For those that do, find their eigenvalues and a basis of eigenvactors. (Hint: Use geometry, not matrices.) (a) R_θ, (b) S_ℓ (ℓ – a line through O), (c) J_r (dilation with scale r), (d) P_U.
11. Find a condition on the numbers a, b, c, d that implies that the matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ has two distinct real eigenvalues
12. Let $A = PMP^{-1}$. Prove that the characteristic polynomials of A and of M are equal (hint: formulae from excercise 1 may prove helpful). Deduce that if M has two distinct eigenvalues λ, μ, then $\det(M) = \lambda\mu$, $\mathrm{tr}(M) = \lambda + \mu$. Is it necessary to assume $\lambda \neq \mu$?
13. Suppose that a linear transformation E satisfies $E \circ E = E$. What are the possible eigenvalues of E? Could E be different from the identity and from the zero map?
14. Show that if $A = PDP^{-1}$ (where D is diagonal), then the columns of P are eigenvectors of A.
15. Suppose that $U, W \in \mathbf{R}^2$ are linearly independentand that the matrix A satisfies $AU = 5U + 4W$, $AW = 3U + 2W$. Prove that there exists an invertible matrix P, such that $A = P\left(\begin{smallmatrix} 5 & 3 \\ 4 & 2 \end{smallmatrix}\right)P^{-1}$. Generalize.
16. Suppose that A—a linear trnsformation of the plane—has two distinct eigenvalues λ, μ. Prove that A preserves orientation iff $\lambda\mu > 0$, and A changes orientation iff $\lambda\mu < 0$. Deduce that S_ℓ (where ℓ is a line through 0) changes orientation.

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17. Suppose that $\det(F - x \cdot Id)$ has a double root λ, and yet F is not a dilation. Prove that there exist two linearly independent vectors U, W, such that $F(U) = \lambda U$, $F(W) = U + \lambda W$. Does F diagonalize? If not, is there a subsitute for diagonalization? (Hint: problem 16)
18. Prove that the following is true: $(\forall A \in M_{2\times2}(\mathbf{R}))((\exists\lambda, \mu \in \mathbf{R})(\exists P \in M_{2\times2}(\mathbf{R}))(A = P\left(\begin{smallmatrix} \lambda & 0 \\ 0 & \mu \end{smallmatrix}\right)P^{-1}) \iff ((\exists\lambda \in \mathbf{R})(A = \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda \end{smallmatrix}\right)) \vee (\exists a, b \in \mathbf{R})(a \neq b \wedge \det(A - aI) = 0 \wedge \det(A - bI) = 0)))$.
19. Let $M = \begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix}$, and let U be an arbitrary vector. Investigate the rate of growth of M^nU as n goes to infinity. (Prove that there exists a constant C that depends (how?) on U, such that the limit $\lim_{n\to+\infty} \frac{\|M^nU\|}{C^n}$ is a positive finite number.)

Seminar material:

20. The Fibonacci sequence is defined recursively: $f_0 = 1$, $f_1 = 1$, $f_{n+1} = f_n + f_{n-1}$ for $n \geq 1$. Derive an explicit formula for its nth term along the following lines:
    a) Put $X_n = \binom{f_n}{f_{n+1}}$, $M = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$. Prove $X_0 = \binom{1}{1}$, $X_{n+1} = MX_n$, $X_n = M^n X_0$.
    b) Diagonalize M: find the eigenvalues and the eigenvaectors, express M as PDP^{-1} with diagonal D.
    c) Find a formula for M^n; deduce a formula for f_n.
    d) What will change if $f_0 = 7$, $f_1 = 3$, $f_{n+1} = f_n + f_{n-1}$ for $n \geq 1$.
21. Investigate for what values of a, b the matrix $\begin{pmatrix} 0 & 1 \\ b & a \end{pmatrix}$ has two distinct eigenvalues. For what values of a, b can you find an explicit formula for the nth term of the sequence defined by $f_{n+1} = af_n + bf_{n-1}$ $(n \geq 1)$, with given f_0 and f_1. Can you cope with the case $f_0 = 0$, $f_1 = 1$, $f_{n+1} = 2f_n - f_{n-1}$?

Old midterm problems:

1. Find an explicit formula for the nth term of the following sequence: $a_0 = 4$, $a_1 = 1$, $a_{n+1} = -3a_n + 10a_{n-1}$ for $n \geq 1$.
2. Let $A, B, C \in \mathbf{R}^2$. Suppose that $\|A+B+C\| = 9$. Does it imply that at least one of the vectors A, B, C
    a) is not shorter than 3?
    b) is not longer than 3?
3. Let ℓ be the line $x = 1$, and let $F : \mathbf{R}^2 \to \mathbf{R}^2$ be a linear transformation with the following property: if $p \in \ell$, then $F(p) \in \ell$.
    a) Prove that $\binom{0}{1}$ is an eigenvector of F.
    b) Prove that 1 is an eigenvalue of F.
    c) Suppose additionally that the eigenvalue of F corresponding to $\binom{0}{1}$ is different from 1. Prove that there exists $q \in \ell$ such that $F(q) = q$.
4. What is the largest possible length of the vector A, if $|\langle \binom{1}{1}, A \rangle| \leq 1$ and $|\langle \binom{1}{2}, A \rangle| \leq 1$? (Hint: a drawing may help.)
5. Let $A = \begin{pmatrix} 100 & \frac{401}{200} \\ 50 & 1 \end{pmatrix}$. Prove that for every $X \in \mathbf{R}^2$

$$\lim_{n \to +\infty} \det(A^n X, A^{n+1} X) = 0.$$