Linear algebra 1R, problem sheet 5

"Symmetry", unless said otherwise, means axial symmetry with respect to a line passing trough 0.

1. Prove: if λ is a real eigenvalue of some linear isometry, than $\lambda = \pm 1$.
2. "The image of vector $\binom{2}{1}$ via a linear isometry F is vector $\binom{1}{3}$."
   Explain, why the sentence above is false.
3. Find out if there exists linear isometry F, such that $F\binom{1}{1} = \binom{1}{-1}$, $F\binom{1}{-1} = \binom{-1}{1}$?
4. Linear isometry F transforms vector $\binom{1}{0}$ to vector $\binom{-1/2}{\sqrt{3}/2}$ and changes orientation. Find $F\binom{0}{1}$.
5. Check, if matrix is a matrix of an isometry. If yes, find out if it is matrix of a rotation or a symmetry. Find corresponding angle or line. (Hint: if A is a matrix of a symmetry, then equation $AX = X$ gives axis of the symmetry.) (a) $\begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$, (b) $\begin{pmatrix} 3/5 & 4/5 \\ -4/5 & 3/5 \end{pmatrix}$, (c) $\begin{pmatrix} -3/5 & 4/5 \\ 4/5 & -3/5 \end{pmatrix}$, (d) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

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6. Complete (in all possible ways) matrix to an isometry matrix: $\begin{pmatrix} * & 3 \\ * & * \end{pmatrix}$; $\begin{pmatrix} * & * \\ * & -\frac{1}{\sqrt{2}} \end{pmatrix}$; $\begin{pmatrix} * & -\frac{1}{\sqrt{2}} \\ * & -\frac{1}{\sqrt{2}} \end{pmatrix}$.
   Which of obtained isometries preserve orientation?
7. How many linear isometres of $\mathbf{R}^2$ have trace $\sqrt{3}$? Find them all; write down their matrices.
8. Superposition $S_\ell \circ S_k$ is a rotation ($\ell = \{t\binom{1}{1} : t \in \mathbf{R}\}$; $k = \{t\binom{0}{2} : t \in \mathbf{R}\}$). Find out its matrix and corresponding angle.
9. Write as $T_U \circ F$ where F is a linear isometry (a) symmetry with respect to line $x = 1$; (b) rotation (counterclockwise) o by angle $\pi/2$ around point $\binom{0}{1}$.
10. Linear transformation has matrix $\begin{pmatrix} 1 & 3 \\ 3 & 4 \end{pmatrix}^{2003}$. Does it preserve orientation?
11. Prove that superposition of 15 symmetries is not a rotation.
12. Let F be a linear isometry. Prove that F is invertible and F^{-1} also is an isometry.
13. Find matrix of symmetry with respect to line $y = ax$.
14. Find all linear isometries transforming X coordinate axis into itself.
15. Compute using matrices: (a) that $R_\theta \circ S_\ell$ is a symmetry. (b) what is $S_\ell \circ R_\theta$. (c) what is $S_\ell \circ S_k$.
16. Each isometry of plane is of form $T_U \circ A$, where A is a linear isometry. Write in this form isometry $(T_X \circ B) \circ (T_Y \circ C)$ (where B, C are linear isometries).
17. Assume that A, B, C, D as linear transformations of plane. Additionaly it is known that A i C are rotations, $\det(D) < 0$, $\det(A \circ B \circ C \circ D) > 0$ and B is an isometry. Find eigenvalues of B.