Linear algebra 1R, problem sheet 6

1. Check that $(AB)^\top = B^\top A^\top$, $(A^\top)^\top = A$, $(A^{-1})^\top = (A^\top)^{-1}$, $\mathrm{tr}(A^\top) = \mathrm{tr}(A)$, $\det(A^\top) = \det(A)$.
2. Prove that $\langle A^\top X, Y\rangle = \langle X, AY\rangle$.
3. Transform to diagonal form the following symmetric matrices (try to minimize calculations):
$$\begin{pmatrix} 1 & 3/2 \\ 3/2 & 1 \end{pmatrix}, \quad \begin{pmatrix} 4 & \sqrt{2} \\ \sqrt{2} & 3 \end{pmatrix}, \quad \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix}.$$
4. Formulas $x' = x + y$, $y' = x - 2y$ give linear, but not rectangular coordinate system on plane.
   a) show two points $P_1 = \begin{bmatrix} x'_1 \\ y'_1 \end{bmatrix}$, $P_2 = \begin{bmatrix} x'_2 \\ y'_2 \end{bmatrix}$, such that $d(P_1, P_2) \neq \sqrt{(x'_1 - x'_2)^2 + (y'_1 - y'_2)^2}$.
   b) show two vectors $U_1 = \begin{bmatrix} x'_1 \\ y'_1 \end{bmatrix}$, $U_2 = \begin{bmatrix} x'_2 \\ y'_2 \end{bmatrix}$, such that $\langle U_1, U_2\rangle \neq x'_1 x'_2 + y'_1 y'_2$.

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5. Prove that in rectangular linear coordinate system formulas from previous problem are equalities.
6. Let $A = \left(\begin{smallmatrix} 1 & 2 \\ 2 & 5 \end{smallmatrix}\right)$ be a matrix of a linear transformation F, and also a matrix of a quadratic form Q (in the standard coordinate system). We introduce new coordinate system by the formulas $x' = 2x + y$, $y' = x + 2y$. Find matrix of F and matrix of Q in the new coordinate system.
7. Give an example of equation of degree 2 of form $ax^2 + 2bxy + cy^2 = d$ describing (a) pair of lines intersecting in point $\binom{0}{0}$; (b) pair of lines parallel to vector $\binom{1}{2}$; (c) circle passing through point $\binom{3}{4}$; (d) $\emptyset$; (f) hyperbole passing through $\binom{0}{3}$, such that angle between its asymptotes is $\pi/3$.
8. A curve has equation $x^2 + xy + y^2 - 1 = 0$. Write equation of the same curve in coordinate system obtained from standard coordinate system by rotation by $\pi/4$. What curve it is?
9. Find canonical forms of the following curves (you can diagonalize corresponding matrices). Recognize and draw the curves (in original coordinates).
   (a) $x^2 + 3xy + y^2 + 1 = 0$, (b) $4x^2 + 2\sqrt{2}xy + 3y^2 = 1$,
10. Investigate and draw curves:
    (c) $x^2 - 2xy + y^2 = 1$, (d) $x^2 - 3y^2 = 0$, (e) $x^2 + 2xy + y^2 = 0$, (f) $x^2 + xy + y^2 = 0$.
11. Describe solutions of $\lambda x^2 + \mu y^2 = 0$ depending on values of parameters λ, μ.
12. Consider equation $ax^2 + 2bxy + cy^2 = h$. It has canonical form $\lambda(x')^2 + \mu(y')^2 = 1$ or $\lambda(x')^2 + \mu(y')^2 = 0$ in some rectangular coordinate system. Looking at signs of $ac - b^2$, a, h we can say a lot about signs of λ and μ (make a table). Use this method to find out if the following equations represent ellipsis, hyperbole, line, pair of lines, point or empty set.
    (a) $x^2 - 3xy + y^2 = 1$, (b) $25x^2 + 10xy + y^2 = 2$, (c) $x^2 - 2xy - 2y^2 = 0$, (d) $x^2 + 4xy + y^2 = -7$,
    (e) $x^2 - 6xy + 9y^2 = 0$, (f) $3x^2 + 7xy - y^2 = -2$, (g) $7x^2 - 2xy + 2y^2 = 0$, (h) $x^2 + xy = -3$,
13. Let X be an eigenvector of a symmetric matrix S, and Y let be a nonzero vector orthogonal to X. Prove that $SY \perp X$ and that Y is an eigenvector of S.
    Hint: The last point is valid only because we deal with matrices of dimension 2.
14. Assume that eigenvalues of a symmetric matrix M are positive.
    a) Show that for every nonzero vector X we have $\langle MX, X\rangle > 0$.
    b) Prove that if P is an invertible matrix then eigenvalues of $P^\top MP$ are positive.