Linear algebra 1R, problem sheet 7

1. Write as $a+bi$: $\frac{(1+i)(2+i)(3+i)}{1-i}$, $\frac{1+i}{1-i}$, $\frac{1+i}{2-i}$, $\frac{1}{i^5}$, $\frac{1}{(-2+i)(1-3i)}$, $\frac{(4-5i)^2}{(2-3i)^2}$.
2. Write in trigonometric form: -1, $1+i$, $-1-\sqrt{3}i$, $7-7i$, $-5+5\sqrt{3}i$.
3. Compute $\overline{(2+3i)\overline{(7-i)}}$.
4. Solve systems of equations: $(a)\begin{cases} z+iw=1 \\ iz+w=1+i \end{cases}$ $(b)\begin{cases} (1+i)z-iw=3+i \\ (2+i)z+(2-i)w=2i \end{cases}$

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5. Solve (in $\mathbf{C}$):
   (a) $z^2-z+1=0$, (b) $z^2+3z+3-i=0$, (c) $z^2+(2i-1)z+1+5i=0$, (d) $z^2+iz=2$, (e) $2z+\overline{z}=6-5i$.
6. Prove: (a) $|-z|=|z|$, (b) $|z/w|=|z|/|w|$, (c) $|z/|z||=1$, (d) $\mathrm{Re}(iz)=-\mathrm{Im}(z)$, (e) $\mathrm{Im}(iz)=\mathrm{Re}(z)$, (f) $\overline{zw}=\overline{z}\,\overline{w}$, (g) $\overline{z+w}=\overline{z}+\overline{w}$, (h) $|z+w|\leq|z|+|w|$.
7. Arithmetic mean of 150 numbers is 1. Prove that at least one of the numbers has absolute value not smaller than 1.
8. Compute products below using trigonometric form:
   (a) $(1+i)(\sqrt{3}+i)$, (b) $(4+4i)(-3+3i)$, (c) $(10-10\sqrt{3}i)(2-2i)$, (d) $(\sqrt{3}+i)^{30}$.
9. Compute (a) $(1+i)^{1000}$; (b) $(1+\frac{\sqrt{3}}{2}+\frac{i}{2})^{24}$; (c) $\left(\frac{1-i\sqrt{3}}{2}\right)^{129}$.
10. Write $\sin(5\phi)$ in terms of $\sin\phi$ and $\cos\phi$. [Hint: Use de Moivre formula.]
11. Derive formula for trigonometric form of quotient of two complex numbers (with given trigonometric forms). Use it to compute:
    (a) $(2+2i)/(1-i)$, (b) $(1-\sqrt{3}i)/(\sqrt{3}+i)$, (c) $3i/(1+i)$.
12. Draw set $\{\frac{1+it}{1-it} : t\in\mathbf{R}\}$.
13. Using trigonometric form compute and draw:
    (a) roots of degree 3 from $8i$; (b) roots of degree 6 from 27; (c) roots of degree 4 from $-(1/2)-(\sqrt{3}/2)i$; (d) roots of degree 8 form 1.
14. Draw on the plane set determined by equation or inequality:
    (a) $\frac{|z+1|}{|z-i|}=1$; (b) $\frac{|z+1|}{|z-i|}=2$; (c) $|\arg z|<\pi/3$; (d) $3<|z-2+i|<5$; (e) $-1<\mathrm{Re}(iz)<0$.
15. Prove that $\left|\frac{z-i}{z+i}\right|<1 \iff \mathrm{Im}(z)>0$. Give geometric interpretation.

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16. A complex number is called primitive root of 1 of degree n if each root of 1 of degree n is its power. Which roots of 1 of degree (a) 3; (b) 12; (c) 16; are primitive roots of 1 of this degree.
17. Compute sum and product of all roots of 1 of degree n.