Linear algebra 1R, problem sheet 8

1. Write in exponential form: -1, $1+i$, $-1-\sqrt{3}i$, $7-7i$, $-5+5\sqrt{3}i$, $1+\frac{\sqrt{3}}{2}+\frac{i}{2}$.
2. Decompose polynomial $P(z)$ into linear factors over $\mathbf{C}$. Also decompose $P(z)$ into linear and irreducible quadratic factors over $\mathbf{R}$. Use fact that number a is a root of $P(z)$.
   (a) $P(z)=z^3-2z^2-5z+6$, $a=-2$;
   (b) $P(z)=z^4+2z^3+7z^2-18z+26$, $a=-2+3i$;
   (c) $P(z)=z^4-3z^3+3z^2-3z+2$, $a=i$;
   (d) $P(z)=z^5+2z^4+2z^3+10z^2+25z$, $a=1-2i$.
3. Write down polynomial $P(z)$ with real coefficients such that 1, 3, $2+i$ are its roots.
4. Write in Jordan form (if needed over $\mathbf{C}$): $\begin{pmatrix}-1 & 1\\ 1 & -1\end{pmatrix}, \begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}$.

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5. Decompose polynomial $P(z)$ into linear factors over $\mathbf{C}$. Also decompose $P(z)$ into linear and irreducible quadratic factors over $\mathbf{R}$.
   (a) $P(z)=z^6+27$;
   (b) $P(z)=z^4+4z^3+4z^2-4z-5$;
   (c) $P(z)=z^4+4$.
6. Let $M\in M_{2\times 2}(\mathbf{C})$. Prove using Jordan theorem that if $M^{100}=0$, than $M^2=0$.
7. Write in Jordan form over $\mathbf{C}$

$$\begin{pmatrix}3 & -4\\ 2 & -1\end{pmatrix}, \quad \begin{pmatrix}4 & -1\\ 1 & 2\end{pmatrix}, \quad \begin{pmatrix}3 & 1\\ -1 & 5\end{pmatrix}.$$

8. Compute $\left(\begin{smallmatrix}1 & 1\\ -1 & 3\end{smallmatrix}\right)^{50}$.
9. Compute $\left(\begin{smallmatrix}7 & -4\\ 14 & -8\end{smallmatrix}\right)^{64}$. Think how to minimize calculations.
10. Compute $\begin{pmatrix}2\sqrt{3} & -7\\ 1 & -\sqrt{3}\end{pmatrix}^{1232}\begin{pmatrix}1\\ 2\end{pmatrix}$. Think how to minimize calculations.
11. Find all $X\in M_{2\times 2}(\mathbf{R})$ such that $X^2=\begin{pmatrix}6 & 2\\ 3 & 7\end{pmatrix}$.
12. Solve equations: (a) $(z+1)^n-(z-1)^n=0$; (b) $(z+i)^n+(z-i)^n=0$.
13. Prove formula $x^{2n+1}-1=(x-1)\Pi_{k=1}^{n}\left(x^2-2x\cos\frac{2k\pi}{2n+1}+1\right)$. Find similar formula for $x^{2n}-1$.

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14. Prove (for $z\neq 1$) formula $1+z+z^2+\ldots+z^n=\frac{1-z^{n+1}}{1-z}$. What you get when z is root of 1 of degree $n+1$?
15. Compute $(2+i)(5+i)(8+i)$. Derive from this the following formula of Strassnitzky:

$$\frac{\pi}{4}=\operatorname{arctg}\frac{1}{2}+\operatorname{arctg}\frac{1}{5}+\operatorname{arctg}\frac{1}{8}.$$