Linear algebra 1R, problem sheet 9

To save space here and in following problem sheets we write $(x,y,z)^\top$ instead of $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$. We also use word orthogonal instead of perpendicular.

1. Prove that vector product is bilinear and antisymmetric.
2. Check via direct calculation:
   (a) $\langle A\times B, C\rangle = \langle B\times C, A\rangle$; (b) $(A\times B)\perp A$, $(A\times B)\perp B$; (d) $A\times(B\times C)+B\times(C\times A)+C\times(A\times B)=0$.
3. Which of expressions below give a vector, which give a number, and which make no sense (A, B, C are vectors, $\lambda\in\mathbf{R}$):
   (a) $\langle C\times(\lambda B), A\rangle$; (b) $\langle C, (\lambda B)\rangle\times A$; (c) $A\times(\langle C,B\rangle B)$; (d) $\lambda\langle A,B\rangle$; (e) $\lambda(A\times B)$; (f) $\langle A,B\rangle + A\times B$; (g) $\langle A,B\rangle(A\times B)$.
4. Find cosine of the angle between planes $2x+7y-z=5$, $-x+y+3z=7$.
5. Find area of parallelograms spanned by pairs of vectors:
   (a) $(1,-2,4)^\top, (-1,2,3)^\top$; (b) $(-1,0,2)^\top, (0,1,3)^\top$.

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6. Prove via calculation: $\|A\times B\| = \|A\|\cdot\|B\|\cdot\sin(\angle(A,B))$. Hint: square both sides and use formulas $\langle A,B\rangle = \|A\|\cdot\|B\|\cdot\cos(\angle(A,B))$, $\sin^2 a+\cos^2 a = 1$ to eliminate sine.
7. Simplify: (a) $\langle A, A\times C\rangle$; (b) $\langle A\times(B+A\times C), A\rangle$; (c) $\langle A+A\times B, A+B\rangle$;
   (d) $\langle D\times(A+D), (A\times B)\times(C\times A)\rangle$.
8. Check which points below are contained in the plane $X=(1,2,3)^\top+t(5,-7,-2)^\top+s(-4,3,-1)^\top$: $(1,2,3)^\top$; $(7,4,2)^\top$; $(-3,4,3)^\top$; $(-2,3,1)^\top$; $(5,6,7)^\top$; $(5,-6,7)^\top$; $(3,-8,-5)^\top$; $(1,\pi,-\pi)^\top$; $(9,-2,7)^\top$; $(12,3,16)^\top$; $(21,-12,13)^\top$; $(11,23,34)^\top$; $(-3,2,1)^\top$; $(-1,1,1)^\top$; $(10,0,10)^\top$; $(2,2,2)^\top$; $(-7,2,5)^\top$; $(1,-1,0)^\top$; $(3,-7,-4)^\top$; $(1,10,100)^\top$.
9. Transform equation of a plane or a line between parametric and nonparametric forms:
   (a) $X=(0,0,1)^\top+t(1,0,1)^\top+s(1,2,7)^\top$; (b) $x+2y+3z=4$; (c) $X=(1,0,7)^\top+t(2,-1,5)^\top$;
   (d) $\frac{x-1}{2}=\frac{y+2}{3}=\frac{z+5}{-1}$.
10. Let $P=(1,2,3)^\top$, Π be the plane $4x+y-z=2$ and ℓ be the line $\frac{x-1}{2}=\frac{y-3}{7}=\frac{z-(-2)}{3}$. Write down nonparametric equations of
    (a) the plane containing P and parallel to Π;
    (b) the plane containing P and ℓ;
    (c) the plane containing P and a line orthogonal to Π;
    (d) the line containing P and parallel to ℓ;
    (e) the line containing P and parallel to Π and orthogonal to ℓ;
    (f) the plane Π' containing ℓ and such that angle between Π and Π' equals angle between Π and ℓ.
11. Prove: if $A+B+C=0$, then $A\times B = B\times C = C\times A$.
12. Find cosine of the angle between the plane $y-5z-1=0$ and the line $3x+4y=0$, $z=0$.
13. Write down parametric and nonparametric equations of the line which is intersection of the planes $x+y+z=7$ and $x+2y-z=3$.
14. Find nonparametric equation of plane containing line $\frac{x-1}{3}=\frac{-1-2y}{4}=\frac{3z+9}{-6}$ and a line orthogonal to the plane $-x+4y-2z=100$.
15. Find equation of plane containing lines $X=(1,1,3)^\top+t(1,-2,1)^\top$, $\frac{1-x}{-2}=\frac{2y+2}{-8}=\frac{z+3}{2}$.
16. Find equation of plane containing $(2,-1,3)^\top$, $(3,1,2)^\top$ and parallel to vector $(-3,1,4)^\top$.
17. Let ℓ be a line containing $X=(0,3,0)^\top+t(-1,1,2)^\top$. Find a line containing $(1,0,1)^\top$ and intersecting ℓ at right angle.

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18. Prove that lines $X=A+tB$ and $X=C+tD$ are contained in a single plane if and only if $\langle A, B\times D\rangle = \langle C, B\times D\rangle$. Using this condition check if lines $X=(1,1,2)^\top+t(7,1,0)^\top$, $X=(-6,0,2)^\top+t(1,0,1)^\top$ are contained in a single plane. If yes find equation of this plane.