Numerical optimization, Problem sheet 1

1. Convert the following linear programming problems to standard form:

(a)

$$\begin{array}{ll} \text{minimize} & x+y \\ \text{subject to} & x+2y=7 \\ & \text{and } x\geq 1,\ y\geq 2. \end{array}$$

(b)

$$\begin{array}{ll} \text{minimize} & x+2y+3z \\ \text{subject to} & 2\leq x+y\leq 3 \\ & 4\leq x+z\leq 5 \\ & x-y\leq 2 \\ & \text{and } x\geq 0,\ y\geq 0,\ z\geq 0 \end{array}$$

(c)

$$\begin{array}{ll} \text{maximize} & 2x+y \\ \text{subject to} & 1\leq x+y\leq 5 \\ & x-y\leq 4 \end{array}$$

2. Write down two linear programming problems in two variables

(a) one which is in standard form and feasible set is empty

(b) second one such that feasible set is a halfline.

3. A manufacturer wishes to produce an alloy that is, by weight, 30% metal A and 70% metal B. Five alloys are available at various prices as indicated below:

Alloy	1	2	3	4	5
% A	10	25	50	75	95
% B	90	75	50	25	5
Price/kg	20	8	6	4	3

The desired alloy will be produced by combining some of the other alloys. The manufacturer wishes to find the amounts of the various alloys needed and to determine the least expensive combination. Formulate this problem as a linear programming problem.

4. Graph the feasible region of linear programming problems below. Graphically find optimal solution.

(a)

$$\begin{array}{ll} \text{maximize} & 2x+y \\ \text{subject to} & x+y\geq 1 \\ & 3x+4y\geq 12 \\ & x-y\leq 2 \\ & -2x+y\leq 2 \end{array}$$

(b)

$$\begin{array}{ll} \text{maximize} & 40x+50y \\ \text{subject to} & 2x+y\leq 12 \\ & -4x+5y\leq 20 \\ & x+3y\leq 15 \\ & \text{and } x\geq 0,\ y\geq 0 \end{array}$$

5. Let

$$s_1 = x_1+x_2+x_3,$$

$$s_2 = x_1x_2+x_1x_3+x_2x_3,$$

$$s_3 = x_1 x_2 x_3,$$

$$Q = 1 - (s_1^2 - 3s_2 + s_3).$$

Consider boolean formulas in variables $x_1, \ldots, x_k$. Let x_{i+k} be negation of x_i (this is to avoid explicitely writing negations). Given boolean formula

$$B = (x_{j_{1,1}} \vee x_{j_{2,1}} \vee x_{j_{3,1}}) \wedge \cdots \wedge (x_{j_{1,m}} \vee x_{j_{2,m}} \vee x_{j_{3,m}})$$

we buid polynomial f in $y_1, \ldots, y_k$ as

$$f = Q(y_{j_{1,1}}, y_{j_{2,1}}, y_{j_{3,1}}) + \cdots + Q(y_{j_{1,m}}, y_{j_{2,m}}, y_{j_{3,m}})$$

where $y_{i+k} = 1 - y_i$. Let S be unit hypercube, that is set of y such that $0 \le y_i \le 1$ for $i = 1, \ldots, k$. Show that $\min f$ on S is zero if and only if there is substitution of truth values for variables in B which makes B true. You may assume that properties of Q given in the lecture are true, that is $0 \le Q \le 1$ with equality only at vertices.

Remark: When f is as above there exist constant $C > 0$ independent of f such that if $\min f > 0$, then $\min f \ge C$. This means that we can reduce known NP-complete problem (boolean satisfability) to an optimization problem for polynomials using moderate precision (in practice standard machine precision).