Numerical optimization, Problem sheet 11

1. Pre-computation for line searches. For each of the following functions, explain how the computational cost of a line search can be reduced by a pre-computation. Give the cost of the pre-computation, and the cost of evaluating $g(t) = f(x + th)$ and $g'(t)$ with and without the pre-computation.

(a) $f(x) = -\sum_{i=1}^{m} \log(b_i - (a_i, x))$.

(b) $f(x) = \log\left(\sum_{i=1}^{m} \exp((a_i, x) + b_i)\right)$.

(c)

$$f(x) = (Ax - b, (P_0 + x_1 P_1 + \cdots + x_n P_n)^{-1}(Ax - b)),$$

where P_i are symmetric m by m matrices, A is m by n matrix, and domain of f consists of such x that $P_0 + x_1 P_1 + \cdots + x_n P_n$ is strictly positive definite.

2. Bound on the distance from the optimum for self-concordant functions. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex self-concordant function such that $f(x)$ goes to infinity when x goes to boundary of the domain. We use notation from end of lecture 6 and start of lecture 7. Suppose $\lambda(f, x_0) < 1$ and the sublevel set $\{x : f(x) \leq f(x_0)\}$ is closed. Assume that the minimum x_∞ of f is attained and show

$$\|x_\infty - x_0\|_{x_0} \leq \frac{\lambda(f, x_0)}{1 - \lambda(f, x_0)}$$

Hint: Use Lemma 1.6 from lecture 6.

3. Nonquadratic conjugate gradient formulas. Check that with exact line searches both formulas for β_i from notes to lecture 7 section 2.4 for nonquadratic functions when applied to quadratic functions give the same result as earlier quadratic conjugate gradient formula.

4. Rewrite momentum formula written in the notes in term of A_i as matrix transforming pair (x_i, x_{i-1}) into (x_{i+1}, x_i). Check that this matrix has norm bigger than 1.

Remark: This is the reason for linear factor in estimate for convergence rate of momentum for quadratic functions.

5. Consider $f(x) = \frac{1}{2}(x_1^2 + x_2^2)$, $g(x) = x_1 + \frac{1}{2}x_2^2$ both with constraint $x_1 = 0$. Use $h(x) = x_1^2$ as penalty function. Check that when λ goes to infinity in both cases problem with penalty becomes increasingly badly conditioned.