Numerical optimization, Problem sheet 2

1. Convert to standard form and solve using simplex procedure (guess basic solution for standard form):

$$\begin{array}{rl} \text{minimize} & 2x_1 + 4x_2 + x_3 + x_4 \\ \text{subject to} & x_1 + 3x_2 + x_4 \leq 4 \\ & 2x_1 + x_2 \leq 3 \\ & x_2 + 4x_3 + x_4 \leq 3 \\ & \text{and } x \geq 0. \end{array}$$

2. Check that in simplex algoritm we only need to compute x_B, r and $B^{-1}Ae_i$. Justify that we can do this without explictely computing inverse, instead solving linear systems of equations. For some matrices M solving equations like $Mx = z$ is much cheaper than matrix invertion, estimate (in big O terms) cost of equation solving that makes equations solving cheaper than incremental update indicated in the lecture. In particular, if we can solve needed equations in $O(n)$ time, where n is number of equations (and variables), is then approach using equation solving better or worse than incremental update?

3. Consider the integer linear programming problem

$$\begin{array}{rl} \text{maximize} & x + 2y \\ \text{subject to} & 3x + 4y \leq 12 \\ & -x + 2y \leq 2 \\ & \text{and } x \geq 0,\ y \geq 0 \\ & \text{with } x,\ y \text{ integers.} \end{array}$$

(a) The optimal solution to the linear program (without the integer restrictions) is $x = 1.6$, $y = 1.8$. Show that rounding this solution leads to an infeasible solution.

(b) Starting from $x = 0$, $y = 0$, systematically enumerate all possible integer solutions and pick an optimal solution. Is it unique?

(c) Replace the first inequality by $3x+4y \leq 8$; the optimal solution to the linear program (without the integer restrictions) now changes to $x = 0.8$, $y = 1.4$. Show that rounding now leads to a feasible solution.

(d) Enumerate all possible feasible integer solutions to the problem as reformulated in (c) and show that the above integer solution is the optimal integer solution.

(e) For the integer linear programming problem

$$\begin{array}{rl} \text{maximize} & x + 2.5y \\ \text{subject to} & 2x + 4.5y \leq 6 \\ & -x + 2y \leq 4 \\ & \text{and } x \geq 0,\ y \geq 0 \\ & \text{with } x,\ y \text{ integers.} \end{array}$$

the optimal solution is $x = 0$, $y = 4/3$. Rounding this results in the feasible solution $x = 0$, $y = 1$. Show that this is not optimal by finding the optimal feasible integer solution.

4. Consider the linear programming problem in standard form

$$\begin{array}{rl} \text{minimize} & 240x_1 + 300x_2 + 200x_3 \\ \text{subject to} & x_1 + 2x_2 + x_3 - x_4 = 2 \\ & 4x_1 + x_2 + x_3 - x_5 = 4 \\ & 3x_1 + 5x_2 + x_3 - x_6 = 3 \\ & \text{and } x_i \geq 0. \end{array}$$

Add three additional variables so that new problem is in standard form, in optimal solution new variables are zero and other give optimal solution to the original problem and the new problem has obvious basic feasible solution.

5 Midpoint convexity. A set C is midpoint convex if whenever two points a, b are in C, the midpoint $(a + b)/2$ is in C. Check that convex set is midpoint convex and that set of rational numbers $\mathbb{Q}$ is midpoint convex but not convex. However, prove that if C is closed and midpoint convex, then C is convex.