Numerical optimization, Problem sheet 3

1. Midpoint convexity. A set C is midpoint convex if whenever two points a, b are in C, the midpoint $(a+b)/2$ is in C. Check that convex set is midpoint convex and that set of rational numbers $\mathbb{Q}$ is midpoint convex but not convex. However, prove that if C is closed and midpoint convex, then C is convex.

2. Which of the following sets are convex?

(a) A slab, i.e., a set of the form $\{x \in \mathbb{R}^n : \alpha \leq \langle a, x\rangle \leq \beta\}$.

(b) A wedge, i.e., $\{x \in \mathbb{R}^n : \langle a_1, x\rangle \leq b_1, \langle a_2, x\rangle \leq b_2\}$.

(c) The set of points closer to a given point than a given set, i.e.,

$$\{x : \|x - x_0\|_2 \leq \|x - y\|_2 \quad \text{for all } y \in S\}$$

where $S \subset \mathbb{R}^n$.

(d) The set of points closer to one set than another, i.e., $\{x : \mathrm{dist}(x, S) \leq \mathrm{dist}(x, T)\}$, where $S, T \subset \mathbb{R}^n$, and $\mathrm{dist}(x, S) = \inf_{z \in S} \|xz\|_2$.

(e) The set $\{x : x + S_1 \subset S_2\}$, where $S_1, S_2 \subset \mathbb{R}^n$ with S_2 convex.

(f) The set of points whose distance to a does not exceed a fixed fraction θ of the distance to b, i.e., the set $\{x : \|x - a\|_2 \leq \theta \|x - b\|_2\}$. You can assume $a \neq b$ and $0 \leq \theta \leq 1$.

3. Support function. The support function of a set $C \subset \mathbb{R}^n$ is defined as

$$S_C(y) = \sup_{x \in C} \langle x, y\rangle.$$

(We allow $S_C(y)$ to take on the value ∞). Suppose that C and D are closed convex sets in $\mathbb{R}^n$. Show that $C = D$ if and only if their support functions are equal. Hint: use separating hyperplane.

4. For each of the following functions determine whether it is convex or concave.

(a) $f(x) = x \exp(x)$ on $(0, \infty)$

(b) $f(x_1, x_2) = x_1 x_2$ on $(0, \infty)^2$.

(c) $f(x_1, x_2) = 1/(x_1 x_2)$ on $(0, \infty)^2$.

(d) $f(x_1, x_2) = x_1/x_2$ on $(0, \infty)^2$.

(e) $f(x_1, x_2) = x_1^2/x_2$ on $\mathbb{R} \times (0, \infty)$.

(f) $f(x_1, x_2) = x^\alpha x_2^{1-\alpha}$, where $0 \leq \alpha \leq 1$, on $(0, \infty)^2$.

5. Convex hull or envelope of a function. The convex hull or convex envelope of a function $f : \mathbb{R}^n \to \mathbb{R}$ is defined as

$$g(x) = \inf\{t : (x, t) \in \mathrm{conv}(\mathrm{epi}(f))\}.$$

Geometrically, the epigraph of g is the convex hull of the epigraph of f. Show that g is the largest convex underestimator of f. In other words, show that if h is convex and satisfies $h(x) \leq f(x)$ for all x, then $h(x) \leq g(x)$ for all x.