Numerical optimization, Problem sheet 4

1. Given three points x_1, x_2, x_3 on the plane consider problem of minimizing sum of distances to x_1, x_2, x_3. Justify that this is convex optimization problem. Show that minimum is attained either at one of x_i, or that angles between x_i viewed from optimal point p are equal to $2\pi/3$ (such point is called Torricelli point). Show that minimum is at one of x_i only when Torricelli point does not exist. Show by example that both cases can happen.

2. For $i = 1, \dots, m$ let $\alpha_i \in \mathbb{R}^n$ and $c_i > 0$. Put $f = \sum_{i=1}^m c_i \exp(\langle \alpha_i, x \rangle)$. Show that $\log(f)$ is convex. Hint: compute second derivative and write numerator as sum of exponentials with coefficients. Use inequality between arithmetic and geometric mean to show that coefficients are positive.

3. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a differentiable function. Assume that y is a local minimum of f along every line that passes trough y, that is for each $h \in \mathbb{R}^n$

$$g(s) = f(y + sh)$$

has local minimum at $s = 0$. Show that $f'(y) = 0$. Check that $f(x_1, x_2) = (x_1 - x_2^2)(x_1 - 2x_2^2)$ satisfies condition above with $y = (0, 0)$, but $(0, 0)$ is not a local minimum of f.

4. Consider problem of minimizing quadratic function $\langle Ax, x \rangle + \langle b, x \rangle + c$. Show that if A is not positive definite (that is there exist x such that $\langle Ax, x \rangle < 0$), then the function is unbounded from below. Show that if optimality condition $2Ax = b$ is not solvable, then the function is unbounded from below.

5. Verify expression for i-th iterate of gradent descent in example $\frac{1}{2}(x_1^2 + \gamma x_2^2)$ given in the lecture.