Numerical optimization, Problem sheet 5

1. Let $f(x) = \sum_{i=1}^{n} \cosh(x_i)$. Consider f on box $-a_i \leq x_i \leq a_i$. Find simple formula (in term of a_i-s) giving optimal values of m and M such that $mI \leq f''(x) \leq mI$ for all x in our box.

2. Let V be the space of complex trigonometric polynomials in one variable of degree at most n, that is each $f \in V$ can be written as $f(x) = \sum_{j=-n}^{n} c_j \exp(ijx)$ with complex c_j. For $f_0 \in V$ consider problem of minimizing $\phi(f) = \|f' + f - f_0\|^2$ where $\|f\|^2 = \sum_{j=-n}^{n} |c_j|^2$. Compute eigenvalues of ϕ''.
Hint: $e_j = \exp(ijx)$ are complex orthonormal basis and $Af = f' + f$ has diagonal matrix in this basis. Use this to write ϕ in simple form in terms of absolute values of complex numbers. Then use fact that for complex number $x = s + it$ absolute value is written as $|x|^2 = s^2 + t^2$.

3. The pure Newton method, that is Newton method with fixed step size $\alpha = 1$ can diverge if the initial point is not close to x_∞. Justify facts about two examples below:
(a) $f(x) = \log(e^x + e^{-x})$ has a unique minimizer $x_\infty = 0$, but there exists t such that if $|x_n| > t$, then $|x_{n+1}| > |x_n|$, so method diverges.
(b) $f(x) = -\log(x) + x$ has a unique minimizer $x_\infty = 1$. There exists t such that for $x_0 < t$ pure Newton method is convergent, but for $x_0 > t$ pure Newton method produces points not in the domain of f.
Check that both functions are convex.

4. Cholesky method. Assume that $A = U^T U$ where U is real upper triangular matrix. Justify that A is positive definite. Find simple formula giving first row of U in terms of elements of A. Explain how this formula leads to algorithm computing U for given A.

5. Verify example from the lecture where

$$f(s) = \frac{1}{2}\left(s_1^2 + s_n^2 + \sum_{i=1}^{n-1}(s_{i+1} - s_i)^2\right) - s_1.$$

(a) show that

$$s_i = 1 - \frac{i}{n+1}$$

is optimal point,
(b) justify that when $x_i \in V_i$, then $f'(x_i) \in V_{i+1}$.