Numerical optimization, Problem sheet 6

1. Check that for $f(x) = \frac{1}{2}(x_1^2 + \frac{1}{2}x_2^2)$ gradient descent with exact line search started in point such that $x_2 > 0$ will stay in points such that $x_2 > 0$. Let $g(x) = f(x)$ for $x_2 \geq 0$ and $f(x) = \frac{1}{2}(x_1^2 - \frac{1}{2}x_2^2)$ for $x_2 < 0$. Justify that g has Lipschitz continuous gradient. Consider gradient descent with line search which chooses local minimum on line closest to starting point. Use first part of the problem to show that such gradient descent started at point such that $x_2 > 0$ will produce the same sequence of approximations for f and g.
Hint: In first part look at derivative of f restricted to search line at intersection with line $x_2 = 0$.

2. Assume that $f : \mathbb{R}^2 \mapsto \mathbb{R}$ has two continuous derivatives and that x_0 is nondegenerate local minimum of f, that is $\nabla^2 f(x_0) \geq mI$ with $m \in \mathbb{R}$ and $m > 0$. Let $g : \mathbb{R}^2 \mapsto \mathbb{R}$ have two continuous derivatives. Show that there exists $\delta, \varepsilon > 0$ such that for all $t \in (-\varepsilon, \varepsilon)$ function $f(x) + tg(x)$ has unique local minimum x_t in ball of radius δ centered at x_0 and we have

$$x_t = x_0 - t(\nabla^2 f(x_0))\nabla g(x_0) + o(t).$$

Hint: This follows from properties of Newton method. Alternativly, you can use implict function theorem.

3. Which of function below are self-concordant? Justify.
(a) $\cosh(x)$ on $\mathbb{R}$.
(b) $f(x, y) = -\log(y^2 - \|x\|_2^2)$ on $\{(x, y) : \|x\|_2 < y\}$.
(c) $f(x, y) = -2\log(y) - \log(y^{2/p} - x^2)$, with $p \geq 1$, on $\{(x, y) \in \mathbb{R}^2 : |x|^p < y\}$.
(d) $f(x, y) = -\log(y) - \log(\log(y) - x)$ on $\{(x, y) : e^x < y\}$.

4. Inverse barrier.
(a) Show that $f(x) = 1/x$ with domain $(0, 8/9)$ is self-concordant.
(b) Show that the function

$$f(x) = c\sum_{i=1}^{m}(b_i\langle a_i, x\rangle)^{-1}$$

with $\mathrm{dom}(f) = \{x \in \mathbb{R}^n : \langle a_i, x\rangle < b_i, i = 1, \ldots, m\}$, is self-concordant if

$$c > (9/8)\max_i \sup_{x\in\mathrm{dom}(f)} (b_i - \langle a_i, x\rangle).$$

5. Assume that f has 3 continuous derivatives. Justify that if f attains minimal value at x_∞ and $f''(x_\infty)$ is strictly positive definite, then there exist positive c and a convex neighbourhood U of x_∞ such that cf is self-concordant in U.