**Numerical optimization, Problem sheet 7**

**1**. Let $A$ be a positive definite matrix and $x$ be arbitrary vector. Show that $x$ can be written as $x = \sum y_i$ where each $y_i$ is an eigenvector of $A$ corresponding to eigenvalue $\lambda_i$ and $\lambda_i$ are all different. Let

$$f(x) = \frac{1}{2}\langle Ax, x\rangle - \langle b, x\rangle$$

with positive definite $A$ and write $x_0 - x_\infty$ (where $x_\infty$ is optimal point) as above. Let $W$ be space spanned by $y_i$. Justify that $x_i$ produced by conjugate gradient method stay in $W + x_0$.

Hint: for first part just join similar terms in usual eigenvalue expansion.

**2**. Consider a method where the first iteration is a steepest descent iteration with exact line search and subsequent iterations have form

$$x_{i+1} = x_i - \alpha_i f'(x_i) - \beta_i(x_i - x_{i-1})$$

where $\alpha_i$ and $\beta_i$ are obtained by two dimensional minimization (that is they minimize value of $f$ on corresponding plane). Show that if $f$ is a quadratic function than this method is the same as conjugate gradient method.

**3**. Explicitly find symmetric rank 1 update satisfying quasi-Newton equations, that is find vector $v$ and scalar $a$ such that

$$U_i(x) = a\langle v, x\rangle v$$

and $S_{i+1} = S_i + U_i$ satisfies quasi-Newton equation.

**4**. Justify that if $f$ is strongly convex ($mI \leq \nabla^2 f$ for some $m > 0$), then in notation from the notes about quasi-Newton method $\langle p_i, q_i\rangle > 0$. Show by example that this may fail for nonconvex $f$.

**5**. Verify that BFGS formula given in part about LBFGS agrees with BFGS formula for $S_i$ given earlier.