NUMERICAL OPTIMIZATION

Sheet 1: Formulating LPs

EXERCISE ONE Formulate the following as a linear program. Use graphical methods (or guesswork) to find the optimal solution.

Ingredients:

	Pizza	Lasagne	available
Tomatoes	2	3	18
Cheese	4	3	24

Profit: Pizza 16 PLN, Lasagne 14 PLN

Task: Determine optimal producible number of pizza and lasagne to maximize total profit.

EXERCISE TWO **Part 1.** Suppose we have a system of linear inequalities that also contains sharp inequalities. One that may look like this:

$$
\begin{aligned}
5x + 3y &\leq 8 \\
2x - 5z &< -3 \\
6x + 5y + 2w &= 5 \\
3z + 2w &> 5 \\
x, y, z, w &\geq 0
\end{aligned}
$$

Is there a way to check if this system has a feasible solution using a linear program?

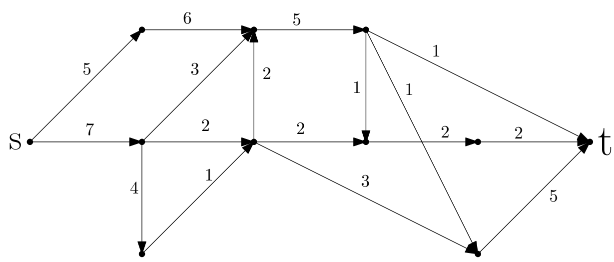
Part 2. Does this mean that linear programming allows strict inequalities? Not really. As a strange example, construct a "linear program with a strict ineqality" that satisfies the following:

- There is a simple finite upper bound on its optimum value;
- There is a feasible solution;
- There is no optimal solution.

This may not happen for a linear program – for a bounded LP, once there exists a feasible solution, there exists also an optimal solution.

EXERCISE THREE Given a graph with directed edges, weights on the edges and two special vertices s and t, we say a *cut* is any collection of edges $S \subseteq E$ whose removal disconnects all directed paths from s to t.

Naturally, since E is a valid cut, we look for the cut with minimum total weight. Using simple logical arguments, can you find the minimum cut on the following directed graph from the lecture?



Exercise four Convert the following linear programming problems to the equational form:

1.

$$\begin{aligned} \text{minimize} \quad & x + y \\ \text{subject to} \quad & x + 2y = 7 \\ & \text{and } x \geq 1, y \geq 2 \end{aligned}$$

2.

$$\begin{aligned} \text{minimize} \quad & x + 2y + 3z \\ \text{subject to} \quad & 2 \leq x + y \leq 3 \\ & 4 \leq x + z \leq 5 \\ & x - y \leq 2 \\ & \text{and } x \geq 0, y \geq 0, z \geq 0 \end{aligned}$$

3.

$$\begin{aligned} \text{maximize} \quad & 2x + y \\ \text{subject to} \quad & 1 \leq x + y \leq 5 \\ & x - y \leq 4 \end{aligned}$$

Exercise five Suppose we have a real matrix A and appropriate vectors b, c. From those, we can build the following integer program C:

$$\begin{aligned} \max c^T x \\ Ax \leq b \\ x \in \{0,1\}^n \end{aligned}$$

Using the same givens, we could also build a linear program L:

$$\begin{aligned} \max c^T x \\ Ax \leq b \\ x \in [0,1]^n \end{aligned}$$

Assume that both programs have a solution. Suppose that we pick one optimal solution of the integer program and call it x^*_C, and we also pick one optimal solution of the linear program, denoting it x^*_L. Prove the following inequality:

$$c^T x^*_C \leq c^T x^*_L.$$

Exercise six Let us consider the following NP-hard problem, called Weighted Vertex Cover:

Input: Undirected graph G with non-negative real weights on the vertices, given by a weight function $w : V(G) \to \mathbb{R}^+_0$.

Goal: To find a subset S of vertices such that each edge $e \in E(G)$ has at least one endpoint in S. (We say that a vertex *covers* an edge, so if we cover all edges, we get a vertex cover.)

From all such subsets S we look for the one with *minimum weight* , i.e. minimum $\sum_{s \in S} w(s)$.

Suggest an integer program with variables $x \in \{0,1\}$ that finds the optimal solution of Weighted Vertex Cover.

Exercise seven Let us now make use of the previous two exercises and figure out a polynomial-time algorithm that finds a 2-approximation of the problem Weighted Vertex Cover.

If you have not yet encountered a 2-approximation algorithm for a minimization problem: The goal is to find a rounding procedure that takes a solution to a linear program and returns a feasible solution x_R for the *integer* linear program, for which it holds that x_R is at most twice as large as the optimum solution for the integer program.