NUMERICAL OPTIMIZATION

Sheet 2: Basic feasible solutions

Exercise one

1. Suppose we have an LP in the form $\max c^T x, Ax \le b, x \ge 0$. We know that for the *equational* form, we can assume that the rows of A are linearly independent. Does this claim also hold for the system $Ax \le b$? Argue why the same idea goes through – or why it fails.
2. We have had a definition of a *basic feasible solution* for the equational form $\max c^T x, Ax = b, x \ge 0$. Naturally, we can say a point is a basic feasible solution for the form $\max c^T x, Ax \le b, x \ge 0$ if it is a basic feasible solution after the conversion to the equational form. Can you reformulate this definition not to refer to the equational form?

Exercise two

Decide if the point $v = (1,1,1)$ is a basic feasible solution of a polytope defined by the following system of inequalities:

a

$$\begin{pmatrix} -1 & -6 & 1 \\ -1 & -2 & 7 \\ 0 & 3 & -10 \\ 1 & 6 & -1 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \le \begin{pmatrix} -6 \\ 5 \\ -7 \\ 6 \end{pmatrix}$$

b

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \le \begin{pmatrix} 1 \\ 1 \\ 3 \\ 2 \end{pmatrix}$$

Exercise three

Apply "brute force" to find all basic feasible solutions of a polytope defined as follows:

$$\begin{aligned} 2x_1 + x_2 + x_3 &\le 14 \\ 2x_1 + 5x_2 + 5x_3 &\le 30 \\ x_1 &\ge 0 \\ x_2 &\ge 0 \\ x_3 &\ge 0 \end{aligned}$$

By brute force we mean checking all the d-tuples of inequalities.

Exercise four

Transform the polytope into the standard equational form:

$$\begin{aligned} x_1 + x_2 &\le 3 \\ x_2 + x_3 &\le 12 \\ x_1 + 3x_2 - x_4 &\ge -7 \\ x_5 &\ge 6 \\ x_2 + x_5 &\le 13 \\ x_1, x_2, x_3, x_4, x_5 &\ge 0 \end{aligned}$$

Additionally, using guesswork, find a basic feasible solution for the equational form.

Exercise five Josef K. got an exercise at his Numerical Optimization class:

Design an integer linear program for the travelling salesman problem: For a given graph with distances $G = (V, E, f)$, where $f : E \to \mathbb{R}_0^+$, find a Hamiltonian cycle with the shortest length.

He suggests the following:

"For every edge uv we have a variable $x_{uv} \in \{0, 1\}$, the target function is $\min \sum_{uv \in E} f(uv) x_{uv}$ and for every vertex u we create a condition of the form $\sum_{i | ui \in E} x_{ui} = 2$."

Prove that Josef K. got the right solution – or prove him wrong.

Exercise six A leftover from Lecture 1. Let

$$\begin{aligned} s_1 &= x_1 + x_2 + x_3, \\ s_2 &= x_1x_2 + x_1x_3 + x_2x_3, \\ s_3 &= x_1x_2x_3, \\ Q &= 1 - \left(s_1^2 - 3s_2 + s_3\right). \end{aligned}$$

Consider boolean formulas in variables $x_1, \ldots, x_k$. Let x_{i+k} be negation of x_i (this is to avoid explicitly writing negations). Given boolean formula

$$B = \left(x_{j_{1,1}} \vee x_{j_{2,1}} \vee x_{j_{3,1}}\right) \wedge \cdots \wedge \left(x_{j_{1,m}} \vee x_{j_{2,m}} \vee x_{j_{3,m}}\right)$$

we buid polynomial f in $y_1, \ldots, y_k$ as

$$f = Q\left(y_{j_{1,1}}, y_{j_{2,1}}, y_{j_{3,1}}\right) + \cdots + Q\left(y_{j_{1,m}}, y_{j_{2,m}}, y_{j_{3,m}}\right)$$

where $y_{i+k} = 1 - y_i$. Let S be unit hypercube, that is set of y such that $0 \le y_i \le 1$ for $i = 1, \ldots, k$. Show that $\min f$ on S is zero if and only if there is substitution of truth values for variables in B which makes B true. You may assume that properties of Q given in the lecture are true, that is $0 \le Q \le 1$ with equality only at vertices.

Remark: When f is as above, there exists a constant $C > 0$ independent of f such that if $\min f > 0$, then $\min f \ge C$. This means that we can reduce a well-known NP-complete problem (boolean satisfability) to an optimization problem for polynomials using moderate precision (in practice standard machine precision).