NUMERICAL OPTIMIZATION

Sheet 3: The simplex method

Exercise one Suppose that we are given the following problem:

$$\begin{aligned} \max x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 \\ x_1 - x_5 + x_6 = 20 \\ x_1 + x_3 + x_7 = 30 \\ x_1 + x_2 + x_4 + x_8 = 10 \\ x_2 - x_3 - x_4 + x_5 + x_9 = 1 \\ x_1, x_2, \ldots, x_9 \geq 0 \end{aligned}$$

and an initial basic solution $(0,0,0,0,0,20,30,10,1)$. Execute one step of the simplex algorithm. Which variable did you pick for your step and why?

Exercise two Solve the following problem by the simplex method, executing all the steps:

$$\begin{aligned} \max 3x_1 + 2x_2 + 4x_3 \\ x_1 + x_2 + 2x_3 \leq 4 \\ 2x_1 + 3x_3 \leq 5 \\ 2x_1 + x_2 + 3x_3 \leq 7 \\ x_1, x_2, x_3 \geq 0 \end{aligned}$$

Exercise three Find any basic feasible solution (a starting tableau for the simplex method) for the following linear program:

$$\begin{aligned} \max 4x_2 - x_4 \\ 3x_1 + x_2 - 2x_4 = 5 \\ -x_2 + x_3 = -2 \\ -2x_1 + 8x_2 + x_3 = 2 \\ x_1, x_2, x_3, x_4 \geq 0 \end{aligned}$$

Note: You can apply the direct simplex method approach or apply some math tricks to get the solution more directly. The choice is up to you.

Exercise four Solve the following linear program using the simplex method:

$$\begin{aligned} \max 9x_1 + 5x_2 + 4x_3 + x_4 \\ 2x_1 + x_2 + x_3 + 2x_4 \leq 2 \\ 8x_1 + 4x_2 - 2x_3 - x_4 \geq 10 \\ 4x_1 + 7x_2 + 2x_3 + x_4 \leq 4 \\ x_1, x_2, x_3, x_4 \geq 0 \end{aligned}$$

Please do not use any "human" shortcuts in your solution. Try to behave like the simplex method implemented by a computer, and write down every step of the process. The pivot selection rule is up to you.

EXERCISE FIVE In the following two exercises, we will play with the *Klee-Minty cube*. In three dimensions, this is the following LP:

$$\begin{aligned} \max 9x_1 + 3x_2 + x_3 & \\ x_1 &\leq 1 \\ 6x_1 + x_2 &\leq 9 \\ 18x_1 + 6x_2 + x_3 &\leq 81 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

Starting from the basic feasible solution $(0, 0, 0, 1, 9, 81)$, can you find the smallest number of pivot steps (steps of the simplex method) that lead to the optimum solution?

EXERCISE SIX Taking the linear program from the last exercise, and starting again at $(0, 0, 0, 1, 9, 81)$, what is the *largest* number of pivot steps (steps of the simplex method) – without any cycling – that lead to the optimum solution?