NUMERICAL OPTIMIZATION

Sheet 4: Convexity of sets and functions

Let us first recall the definition of a basic feasible solution for general polytopes:

D(Basic feasible solution**):** For an polytope P in $\mathbb{R}^n$, we say a point p is a basic feasible solution if $p \in P$ and the matrix A_p has rank n, where A_p is the submatrix consisting of rows of A which are satisfied with equality.

D(Vertex**):** For a polytope P, we say a point $v \in P$ is a vertex if v *cannot* be written as a convex combination of any two points in P.

EXERCISE ONE First, prove that any point p which is *not* a vertex is not a basic feasible solution.

Hint: Using the fact that p is not a vertex, create a vector y such that $A_p y = 0$ and $y \neq 0$. Why does it imply something about rank?

EXERCISE TWO Next, prove that any point which is not a basic feasible solution is not a vertex.

Hint: Proceed in reverse to the following exercise. Argue that we know there exists $y, y \neq 0$ such that $A_p y = 0$. In fact, we can find y' such that $Ay' = 0$. Finally, argue that $p + \epsilon y'$ and $p - \epsilon y'$ can form p as their convex combination – and that both $p \pm \epsilon y'$ lie within P.

Combining the last two exercises, we reach the general statement of the theorem:

T:For a polytope, a point is a vertex if and only if it is a basic feasible solution.

EXERCISE THREE Which of the following sets are convex?

1. A slab, i.e., a set of the form $\{x \in \mathbb{R}^n : \alpha \leq \langle a, x\rangle \leq \beta\}$.
2. A wedge, i.e., $\{x \in \mathbb{R}^n : \langle a_1, x\rangle \leq b_1, \langle a_2, x\rangle \leq b_2\}$.
3. The set of points closer to a given point than a given set, i.e.,
$$\{x : \|x - x_0\|_2 \leq \|x - y\|_2 \text{ for all } y \in S\}, \qquad \text{where } S \subset \mathbb{R}^n.$$
4. The set of points closer to one set than another, i.e., $\{x : \text{dist}(x, S) \leq \text{dist}(x, T)\}$, where $S, T \subset \mathbb{R}^n$, and $\text{dist}(x, S) = \inf_{z \in S} \|xz\|_2$.
5. The set $\{x : x + S_1 \subset S_2\}$, where $S_1, S_2 \subset \mathbb{R}^n$ with S_2 convex.
6. The set of points whose distance to a does not exceed a fixed fraction θ of the distance to b, i.e., the set $\{x : \|x - a\|_2 \leq \theta\|x - b\|_2\}$. You can assume $a \neq b$ and $0 \leq \theta \leq 1$.

EXERCISE FOUR A set C is midpoint convex if whenever two points a, b are in C, the midpoint $(a + b)/2$ is in C. Check that a convex set is midpoint convex and that the set of rational numbers $\mathbb{Q}$ is midpoint convex but not convex. However, prove that if C is closed and midpoint convex, then C is convex.

Hints: For the last claim, imagine we could mathematically cheat a little and prove something by induction on the distance between $a, b \in C$. Could you prove it in this simple setting?

To avoid the cheating trick, some basic properties of closed sets and open sets might be useful. I suggest a little bit of Google search.

EXERCISE FIVE For each of the following functions determine whether it is convex, concave, or neither.

1. $f(x) = xe^x$ on $(0, \infty)$
2. $f(x_1, x_2) = x_1 x_2$ on $(0, \infty)^2$.

3. $f(x_1, x_2) = 1/(x_1 x_2)$ on $(0, \infty)^2$.
4. $f(x_1, x_2) = x_1/x_2$ on $(0, \infty)^2$.
5. $f(x_1, x_2) = x_1^2/x_2$ on $\mathbb{R} \times (0, \infty)$.
6. $f(x_1, x_2) = x_1^{\alpha} x_2^{1-\alpha}$, where $0 \leq \alpha \leq 1$, on $(0, \infty)^2$.