Numerical optimization, Problem sheet 6

1. Prove the convexity of $g(x) = \log\left(\sum_{i=1}^{m} e^{x_i}\right)$ by checking the positive semidefiniteness of the Hessian. This function $g(x)$ can be seen as a special instance of the function f from Problem 5.2.

2. Let $f(x) = \sum_{i=1}^{n} \cosh(x_i)$. Consider f on box $-a_i \le x_i \le a_i$. Find simple formula (in term of a_i-s) giving optimal values of m and M such that $mI \le f''(x) \le mI$ for all x in our box.

3. The pure Newton method, that is Newton method with fixed step size $\alpha = 1$ can diverge if the initial point is not close to x_∞. Justify facts about two examples below:

(a) $f(x) = \log(e^x + e^{-x})$ has a unique minimizer $x_\infty = 0$, but there exists t such that if $|x_n| > t$, then $|x_{n+1}| > |x_n|$, so method diverges.

(b) $f(x) = -\log(x) + x$ has a unique minimizer $x_\infty = 1$. There exists t such that for $x_0 < t$ pure Newton method is convergent, but for $x_0 > t$ pure Newton method produces points not in the domain of f.

Check that both functions are convex.

4. Cholesky method. Assume that $A = U^T U$ where U is real upper triangular matrix. Justify that A is positive definite. Find simple formula giving first row of U in terms of elements of A. Explain how this formula leads to algorithm computing U for given A.

Remark: Cholesky method can be used to verify that matrix is positive definite.

5. Verify example from the lecture where

$$f(s) = \frac{1}{2}\left(s_1^2 + s_n^2 + \sum_{i=1}^{n-1}(s_{i+1} - s_i)^2\right) - s_1.$$

(a) show that

$$s_i = 1 - \frac{i}{n+1}$$

is optimal point,

(b) justify that when $x_i \in V_i$, then $f'(x_i) \in V_{i+1}$.