Numerical optimization, Problem sheet 7

1. Which of function below are self-concordant? Justify.
(a) $\cosh(x)$ on $\mathbb{R}$.
(b) $f(x, y) = -\log(y^2 - \|x\|_2^2)$ on $\{(x, y) : \|x\|_2 < y\}$.
(c) $f(x, y) = -2\log(y) - \log(y^{2/p} - x^2)$, with $p \geq 1$, on $\{(x, y) \in \mathbb{R}^2 : |x|^p < y\}$.
(d) $f(x, y) = -\log(y) - \log(\log(y) - x)$ on $\{(x, y) : e^x < y\}$.

2. Inverse barrier.
(a) Show that $f(x) = 1/x$ with domain $(0, 8/9)$ is self-concordant.
(b) Show that the function

$$f(x) = c\sum_{i=1}^{m}(b_i\langle a_i, x\rangle)^{-1}$$

with $\mathrm{dom}(f) = \{x \in \mathbb{R}^n : \langle a_i, x\rangle < b_i, i = 1, \ldots, m\}$, is self-concordant if

$$c > (9/8)\max_i \sup_{x\in\mathrm{dom}(f)} (b_i - \langle a_i, x\rangle).$$

3. Assume that f is a nondegenerate selfconcordant function such that $f(x)$ goes to infinity when x goes to the boundary of domain of f. Assume that f is bounded from below. Show that f attains minimal value.
Hint: First show that $\inf \lambda(f, x) = 0$ using decay estimate for Newton method.

4. Analytic center of linear inequalities with variable bounds. Give the most efficient method for performing one step of Newton method for the function

$$f(x) = -\sum_{i=1}^{n}(\log(1 + x_i) + \log(1 - x_i)) - \sum_{i=1}^{m}\log(b_i - (a_i, x))$$

where b_i are scalars and a_i are vectors, with natural domain, that is where arguments of all logarithms are positive. Distinguish cases when $n \leq m$ and $n \geq m$.

5. Assume that f has 3 continuous derivatives. Justify that if f attains minimal value at x_∞ and $f''(x_\infty)$ is strictly positive definite, then there exist positive c and a convex neighbourhood U of x_∞ such that cf is self-concordant in U.