Numerical optimization, Problem sheet 8

1. Let A be a positive definite matrix and x be arbitrary vector. Show that x can be written as $x = \sum y_i$ where each y_i is an eigenvector of A corresponding to eigenvalue λ_i and λ_i are all different. Let

$$f(x) = \frac{1}{2}\langle Ax, x\rangle - \langle b, x\rangle$$

with positive definite A and write $x_0 - x_\infty$ (where x_∞ is optimal point) as above. Let W be space spanned by y_i. Justify that x_i produced by conjugate gradient method stay in $W + x_0$.

Hint: for first part just join similar terms in usual eigenvalue expansion.

2. Consider a method where the first iteration is a steepest descent iteration with exact line search and subsequent iterations have form

$$x_{i+1} = x_i - \alpha_i f'(x_i) - \beta_i(x_i - x_{i-1})$$

where α_i and β_i are obtained by two dimensional minimization (that is they minimize value of f on corresponding plane). Show that if f is a quadratic function than this method is the same as conjugate gradient method.

3. Nonquadratic conjugate gradient formulas. Check that with exact line searches both formulas for β_i for nonquadratic functions when applied to quadratic functions give the same result as earlier quadratic conjugate gradient formula.

4. Put $p_i = A^i p_0$. Let d_i be A-orthogonal vectors obtained by Gram-Schmidt procedure from p_i. Show that d_i can be obtained using three-term recurrence, that is d_{i+1} can be obtained from Ad_i, d_i and d_{i-1}.

5. Explicitly find symmetric rank 1 update satisfying quasi-Newton equations, that is find vector v and scalar a such that

$$U_i(x) = a\langle v, x\rangle v$$

and $S_{i+1} = S_i + U_i$ satisfies quasi-Newton equation.