Numerical optimization, Additional problem sheet

1. Assume that f is Lipschitz continuous with constant M and that S is closed. Show that for $\lambda > M$ problem of minimizing $f(x) + \lambda d(x, S)$ where $d(x, S)$ is distance from x to S have the same solutions as problem of minimizing $f(x)$ over S.

2. Compute conjugate (Legendre transform) of max function: $f(x) = \max_{i=1,\dots,n} x_i$.

3. Find formula for projection onto $[0, \infty)^n$. Justify.

4. Find formula for projection onto a convex cone C in $\mathbb{R}^2$. Boundary of C consists of two half-lines, write formula in terms of those half-lines. Give geometric illustration of the formula.

5. Directly from definition compute $\mathrm{prox}_f(x)$ for $f(x) = -\log(x)$ and $f(x) = \max(x^3, 0)$.

6. Let C be closed convex set. Directly from definitions check that $I_C^* = S_C$.

7. Fix nonzero $a \in \mathbb{R}^n$. Let $f(x) = \langle a, x\rangle$. Find $\mathrm{prox}_f(z)$. Use result to show that simple iteration of proximal operator (that is sequence $x_{i+1} = \mathrm{prox}_f(x_i)$) does not need to converge.

8. Let

$$f(x) = g(x) + \frac{c}{2}\|x - a\|^2.$$

Show that

$$\mathrm{prox}_{\lambda f}(x) = \mathrm{prox}_{\bar\lambda g}(\frac{\bar\lambda}{\lambda}x + c\bar\lambda a)$$

where $\bar\lambda = \frac{\lambda}{1+c\lambda}$.

Hint: Normalize function to minimize to have equal coefficient before g. Show that after such normalization gradients of quadratic terms are equal.