Generic tuples and free groups

Tomasz Rzepecki
(WIP joint with Itay Kaplan and Nicholas Ramsey)

Uniwersytet Wrocławski

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Fraïssé classes

Definition

Let $\mathcal{C}$ be a class of finitely generated structures in a common language. We say that $\mathcal{C}$ is a Fraïssé class when:

- it is essentially countable (it has countably many elements up to isomorphism),
- it has the hereditary property (HP), i.e. it is closed under isomorphism and under taking a finitely generated substructure,
- it has the joint embedding property (JEP), i.e. for any $A, B \in \mathcal{C}$, there is some $C \in \mathcal{C}$ into which A and B embed,
- it has the amalgamation property (AP), i.e. for any $A \leftarrow C \rightarrow B$, there is some D into which A and B embed, consistently over C.

Fraïssé limits

Fact

Let $\mathcal{C}$ be a Fraïssé class. Then there is a unique countable structure M such that:

- *every f.g. substructure of M is in $\mathcal{C}$,*
- *every $\mathcal{C}$-structure embeds into M (embedding property),*
- *given any $A, B \in \mathcal{C}$ and embeddings $i_A \colon A \to M$, $i_{AB} \colon A \to B$, there is $i_B \colon B \to M$ such that $i_A = i_B i_{AB}$ (extension property).*

Definition

This M is called the Fraïssé limit of $\mathcal{C}$.

Examples

- Finite graphs form a Fraïssé class, the limit is the random (Rado) graph.
- Finite dimensional vector spaces over a fixed countable field are Fraïssé ; the limit is a countably infinite dimensional space.
- Finite linear orders form a Fraïssé class. The limit is isomorphic to $(\mathbf{Q}, <)$.
- Finite Boolean algebras are Fraïssé , the limit is the countable atomless Boolean algebra.
- Finite total cyclic orderings are Fraïssé , and the limit is the countable dense cyclic ordering $\mathbf{Q}/\mathbf{Z}$.
- Finite trees are essentially countable, have HP and JEP, but not AP in the language $\{<\}$, but are Fraïssé in the langugе $\{<, \wedge\}$, and the limit is the generic meet-tree T_∞.
- Finitely generated groups have HP, JEP and AP, but are not essentially countable.

Generic tuples

Definition

Let G be a Polish (i.e. separable, completely metrisable) group.

- We say that $g \in G$ is generic if its conjugacy class is comeagre in G.
- We say that $(g_1, g_2, \ldots, g_n) = \bar{g} \in G^n$ is generic if its diagonal conjugacy class, i.e.

$$\{(g_1^g, g_2^g, \ldots, g_n^g) \mid g \in G\},$$

  is comeagre in G^n.

Remark

By Kuratowski-Ulam, this is equivalent to saying that
for each $i = 1, 2, \ldots, n$, $g_i^{C(g_1) \cap \ldots \cap C(g_{i-1})}$ is comeagre in G

Fact

If $\bar{g}$ is a generic tuple in $G \neq 0$, then it freely generates a subgroup of G.

Ample generics

Definition

Let G be a Polish group. We say that $(g_1, g_2, \dots, g_n) = \bar{g} \in G^n$ is generic if its diagonal conjugacy class is comeagre in G^n.

Definition

We say that G has *ample generics* if for each natural n, it has a generic n-tuple.

Fact

If G has ample generics and H is a separable topological group, then any homomorphism $G \to H$ is continuous.

Corollary

If $G = \mathrm{Aut}(M)$, $H = \mathrm{Aut}(N)$, M, N are ω-categorical and have ample generics, while $G \cong H$ (as groups), then M and N are bi-interpretable (via Ahlbrandt-Ziegler).

Generics and countable structures

- When M is a countable structure, the group $\mathrm{Aut}(M)$ with the pointwise convergence topology is a Polish group, so it makes sense to ask about its generic automorphisms.
- Ivanov and, independently, Kechris-Rosendal, gave a full characterisation of when a Fraïssé limit M has a generic tuple of automorphisms (in terms of the class of partial automorphisms of $\mathcal{C}$-structures).
- In the cases we will consider, $\bar{\sigma} \in \mathrm{Aut}(M)^n$ will be generic exactly when $(M, \bar{\sigma})$ is the limit of $\mathcal{C}^n$ (the class of $\mathcal{C}$-structures with n automorphisms; it is often not a Fraïssé class, but here it will be).

Some examples

Example

- $(\mathbf{Q}, <)$ has a generic automorphism but no generic pair of automorphisms;
- the generic meet-tree $\mathbf{T}_\infty$ has a generic automorphism, but no generic pair of automorphisms;
- pure sets have ample generics;
- the random graph (and its variants) has ample generics;
- vector spaces of countably infinite dimension have ample generics;
- atomless Boolean algebras have ample generics.

Open problems

Question

Suppose M is Fraïssé limit which is unstable and NIP (or: linearly ordered).
Can M have a generic pair of automorphisms?

Question (Two-three question)

Suppose M is a Fraïssé limit. Suppose M has a generic pair of automorphisms. Does it have a generic triple of automorphisms? (More generally, does it have ample generics?)

Question (Square question)

Suppose M is a Fraïssé limit, and suppose σ is a generic automorphism of M.
Is σ^2 necessarily a generic automorphism of M?

(Symmetric) canonical JEP and AP

Definition

We say that $\mathcal{C}$ has *canonical JEP [AP]* if [for any $C \in \mathcal{C}$] there is a functor mapping each pair A, B [/each span $A \leftarrow C \rightarrow B$] $\mathcal{C}$ to $A \otimes B[/A \otimes_C B] \in \mathcal{C}$, admitting natural embeddings $A, B \rightarrow A \otimes B[/A \otimes_C B]$ [which make the two embeddings of C commute].

Definition

We say that $\mathcal{C}$ has *symmetric* (canonical) JEP [AP] if $\bigotimes_{[C]}$ is well-defined.
(In particular, $A \otimes_{[C]} B$ is naturally isomorphic to $B \otimes_{[C]} A$; e.g. $\otimes$ is commutative and associative.)

Example

- Free amalgamation classes, vector spaces (over fixed K), Boolean algebras have symmetric canonical JEP and AP.
- Linear orders have asymmetric canonical JEP and AP.
- Total cyclic orders do not have canonical JEP nor AP.

A question

Question (main question)

Suppose $\mathcal{C}$ is a Fraïssé class with symmetric JEP and AP, M is the limit of $\mathcal{C}$, $\bar{\sigma}$ is a generic n-tuple of automorphisms of M, and $\bar{w}$ is an m-tuple of words in $\bar{\sigma}$ with no algebraic dependencies.
Is $\bar{w}$ a generic m-tuple of automorphisms?

Example

$\bar{\sigma} = \sigma$, $\bar{w} = \sigma^2$ corresponds to the square question.

Example

$\bar{\sigma} = (\sigma_1, \sigma_2)$, $\bar{w} = (\sigma_2, \sigma_1\sigma_2\sigma_1^{-1}, \sigma_1^2)$ corresponds to the two-three question.

A partial result

Question (main question)

Suppose $\mathcal{C}$ is a Fraïssé class with symmetric JEP and AP, M is the limit of $\mathcal{C}$, $\bar{\sigma}$ is a generic n-tuple of automorphisms of M, and $\bar{w}$ is an m-tuple of words in $\bar{\sigma}$ with no algebraic dependencies.
Is $\bar{w}$ a generic m-tuple of automorphisms?

Theorem

Suppose $\mathcal{C}$ is a Fraïssé class with symmetric JEP and AP, EPPA ($+$ a minor technical assumption).
Then if M is the limit of $\mathcal{C}$ and σ is a generic automorphism of M (it always exists), then σ^2 is also a generic automorphism of M.

Examples

Theorem

Suppose $\mathcal{C}$ is a Fraïssé class with symmetric JEP and AP, EPPA (+ a minor technical assumption).
Then if M is the limit of $\mathcal{C}$ and σ is a generic automorphism of M (it always exists), then σ^2 is also a generic automorphism of M.

Some contexts in which the theorem applies:

- free amalgamation classes, e.g.:
  - random graphs,
  - random hypergraphs,
  - K_n-free random graphs, etc.
- vector spaces over a fixed, possibly infinite field.

Technical lemmas

Lemma (embedding lemma, symmetric JEP)

Suppose (A, p_A) is a $\mathcal{C}$-structure with an automorphism. Then there is an $\bar{A} \supseteq A$ admitting an automorphism $p_{\bar{A}}$ such that $p_A \subseteq p_{\bar{A}}^2$.

Lemma (extension lemma, symmetric AP)

Suppose (A, p_A), $(\bar{A}, p_{\bar{A}})$ and (B, p_B) are $\mathcal{C}$-structures with automorphisms such that $(A, p_A) \subseteq (\bar{A}, p_{\bar{A}}^2)$ and $(A, p_A) \subseteq (B, p_B)$. Then there is a $\bar{B} \supseteq B, \bar{A}$ admitting an automorphism $p_{\bar{B}}$ such that $p_B \subseteq p_{\bar{B}}^2$ and $p_{\bar{A}} \subseteq p_{\bar{B}}$.

Corollary (symmetric AP+JEP)

- *If (M, σ) has the embedding property for $\mathcal{C}^1$, then so does (M, σ^2).*
- *If (M, σ) has the extension property for $\mathcal{C}^1$, then so does (M, σ^2).*

Proof of the embedding lemma

Lemma (embedding lemma, symmetric JEP)

Suppose (A, p_A) is a $\mathcal{C}$-structure with an automorphism. Then there is an $\bar{A} \supseteq A$ admitting an automorphism $p_{\bar{A}}$ such that $p_A \subseteq p_{\bar{A}}^2$.

- Take $\bar{A} = A \otimes A$
- Write τ for the automorphism of $A \otimes A$ given by symmetry of $\otimes$.
- We claim that $p_{\bar{A}} = \tau(\mathrm{id}_A \otimes p_A)$ works.
- Indeed, for each $a \in A$, write a_1, a_2 for its two copies in $A \otimes A$ (given by the two embeddings of A).
- Then $\tau(a_1) = a_2$ and $\tau(a_2) = a_1$, so $p_{\bar{A}}(a_1) = \tau((\mathrm{id}_A(a))_1) = \tau(a_1) = a_2$ and $p_{\bar{A}}(a_2) = \tau((p_A(a))_2) = p_A(a)_1$.

Proof of the extension lemma

Lemma (extension lemma, symmetric AP)

Suppose (A, p_A), $(\bar{A}, p_{\bar{A}})$ and (B, p_B) are $\mathcal{C}$-structures with automorphisms such that $(A, p_A) \subseteq (\bar{A}, p^2_{\bar{A}})$ and $(A, p_A) \subseteq (B, p_B)$. Then there is a $\bar{B} \supseteq B, \bar{A}$ admitting an automorphism $p_{\bar{B}}$ such that $p_B \subseteq p^2_{\bar{B}}$ and $p_{\bar{A}} \subseteq p_{\bar{B}}$.

- Take $\bar{B} = (B \otimes_A \bar{A}) \otimes_{\bar{A}} (B \otimes_{p_{\bar{A}}[A]} \bar{A})$
- Write τ for isomorphism $(B \otimes_A \bar{A}) \otimes_{\bar{A}} (B \otimes_{p_{\bar{A}}[A]} \bar{A}) \to (B \otimes_{p_{\bar{A}}[A]} \bar{A}) \otimes_{\bar{A}} (B \otimes_A \bar{A})$ given by symmetry of $\otimes$.
- We claim that $p_{\bar{B}} = \tau((\mathrm{id}_B \otimes p_{\bar{A}}) \otimes (p_B \otimes p_{\bar{A}}))$ works.
- Note that $\mathrm{id}_B \otimes p_{\bar{A}}$ yields an isomorphism $B \otimes_A \bar{A} \to B \otimes_{p_{\bar{A}}[A]} \bar{A}$, while $p_B \otimes p_{\bar{A}}$ yields an isomorphism $B \otimes_{p_{\bar{A}}[A]} \bar{A} \to B \otimes_A \bar{A}$, so $p_{\bar{B}}$ is an automorphism of $\bar{B}$.
- Now, for each $b \in B$, write b_1, b_2 for its copies in $B \otimes_A \bar{A}$ and $B \otimes_{p_{\bar{A}}[A]} \bar{A}$.
- Then as before, $p_{\bar{B}}(b_1) = b_2$ and $p_{\bar{B}}(b_2) = (p_B(b))_1$.

Finishing the theorem

Theorem

Suppose $\mathcal{C}$ is a Fraïssé class with symmetric JEP and AP, EPPA (+ a minor technical assumption).
Then if M is the limit of $\mathcal{C}$ and σ is a generic automorphism of M (it always exists), then σ^2 is also a generic automorphism of M.

Proof.

- The hypotheses imply that $\tau \in \mathrm{Aut}(M)$ is a generic automorphism of M if and only if (M, τ) has the embedding and extension properties for $\mathcal{C}^1$.
- By the lemmas, the embedding and extension properties of (M, σ) with respect to $\mathcal{C}^1$ imply the same for (M, σ^2), so (M, σ^2) is the Fraïssé limit of $\mathcal{C}^1$, and hence σ^2 is generic. □

Going further

Lemma (Stronger embedding lemma, symmetric JEP)

Suppose $\Gamma \leq G$ is a subgroup of finite index.
Suppose $(A, \gamma)_{\gamma \in \Gamma}$ is a $\mathcal{C}$-structure with an action of Γ by automorphisms.
Then we can find $(\bar{A}, g)_{g \in G}$, a $\mathcal{C}$-structure with an action of G by automorphisms, such that $(A, \gamma)_{\gamma} \subseteq (\bar{A}, \gamma)_{\gamma}$.

Proof.

- Consider $\tilde{A} = A \times G/{\sim}$,
  where $(a_1, g_1) \sim (a_2, g_2)$ when $g_2^{-1} g_1 \in \Gamma$ and $g_2^{-1} g_1(a_1) = a_2$.
- Let $\bar{A} = \bigotimes_{[g] \in G/\Gamma} A_{[g]}$, where $A_{[g]} = \{[a, g]_\sim \mid a \in A\}$. Each $h \in G$ gives us a morphism $\sigma_{h,g} \colon A_{[g]} \to A_{[hg]}$, $[a, g]_\sim \mapsto [a, hg]_\sim$.
- These maps combine to $\bigotimes_{[g] \in G/\Gamma} \sigma_{h,g} \colon \bar{A} \to \bigotimes_{[g] \in G/\Gamma} A_{[hg]}$, which we identify with an automorphism of $\bar{A}$ (by symmetry). This yields the G-action. □

Going further

Lemma (Stronger embedding lemma, symmetric JEP)

Suppose $\Gamma \leq G$ is a subgroup of finite index.
Suppose $(A, \gamma)_{\gamma \in \Gamma}$ is a $\mathcal{C}$-structure with an action of Γ by automorphisms.
Then we can find $(\bar{A}, g)_{g \in G}$, a $\mathcal{C}$-structure with an action of G by automorphisms, such that $(A, \gamma)_\gamma \subseteq (\bar{A}, \gamma)_\gamma$.

- To recover the original embedding lemma, consider $G = \mathbf{Z}$ and $\Gamma = 2\mathbf{Z}$.
- When G is a free group of finite rank (e.g. a group generated by a generic tuple) and Γ is finitely generated, then we do not need to assume that Γ has finite index (because then, by Hall's lemma, Γ is a free factor in some $\tilde{\Gamma} \leq G$ of finite index).
- There is an analogous generalisation of the extension lemma.

Another question

Question (main question)

Suppose $\mathcal{C}$ is a Fraïssé class with symmetric JEP and AP, M is the limit of $\mathcal{C}$, $\bar{\sigma}$ is a generic n-tuple of automorphisms of M, and $\bar{w}$ is an m-tuple of words in $\bar{\sigma}$ with no algebraic dependencies.
Is $\bar{w}$ a generic m-tuple of automorphisms?

Remark

If yes, then if $\bar{\sigma}$ is a generic n-tuple, then if $1 \neq w \in F_n$, then $w(\bar{\sigma})$ is a generic automorphism.

Question

Suppose $\bar{\sigma}$ is an n-tuple of automorphisms such that for any $1 \neq w \in F_n$, $w(\bar{\sigma})$ is generic. Does it follow that $\bar{\sigma}$ is generic?

End

Precise statement

Theorem

If $\mathcal{C}$ is a HFG Fraïssé class with EPPA as well as symmetric AP and JEP, M is the Fraïssé limit of $\mathcal{C}$, $\bar{\sigma}$ is a generic n-tuple of automorphisms of M and $\bar{w}$ is an m-tuple freely generating a subgroup of F_n, then $\bar{w}(\bar{\sigma})$ is a generic m-tuple of automorphisms of M.

Here:

- HFG (= hereditarily finitely generated) means that every substructure of a $\mathcal{C}$-structure is finitely generated (so a $\mathcal{C}$-structure),
- EPPA (= extension property for finite automorphisms) means that any finite family of partial automorphisms of a $\mathcal{C}$-structure can be extended to a family of automorphisms of a larger $\mathcal{C}$-structure,
- symmetric AP and JEP are as defined before.

We would like to prove the same without assuming EPPA, which will require working with partial automorphisms and weak (existential) extension property