Model theory, topological dynamics and descriptive set theory

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Things I want to talk about:

- Essential correspondence between model-theoretical properties of first order theories and dynamical properties of related dynamical systems.
- Descriptive-set-theoretic invariants of first order theories: Galois groups, their Borel cardinalities and Polish group extensions.

Type spaces

Recall that if M is a model and x is a tuple of variables, $S_x(M)$ is (equivalently):

- the set of complete types over M in variables x,
- the set of maximal consistent sets of formulas with free variables x and parameters from M,
- the Stone (ultrafilter) space of the boolean algebra of (equivalence classes of) formulas with parameters in M and free variables x.

Dynamical systems

Definition

A *metric dynamical system* (G, X) is a group G acting by homeomorphisms on a compact Polish space X.

Example

Suppose M is a model. For $\sigma \in \mathrm{Aut}(M)$ and $p \in S_x(M)$, write $\sigma \cdot p$ for the complete type consisting of formulas $\varphi(x, \sigma^{-1}(a))$ for $\varphi(x, a) \in p(x)$. This defines an action of $\mathrm{Aut}(M)$ on the space of types $S_x(M)$.

WAP dynamical systems

Definition

We say that a metric dynamical system (G, X) is *weakly almost periodic* (WAP) if for every $f \in C(X)$, the pointwise closure of $G \cdot f \subseteq \mathbf{R}^X$ consists of continuous functions. (Here, $(g \cdot f)(x) = f(g^{-1}x)$.)

Example

The action of $G = \mathbf{Z}$ on $X = \mathbf{Z} \cup \{-\infty, +\infty\}$ by translation is not WAP. Just notice that $\lim_{k \to +\infty} -k + x$ is $+\infty$ for $x = +\infty$, and for all other x, it is $-\infty$.

Fact (WAP testing)

If X is totally disconnected, it is enough to check WAP on characteristic functions of basic clopen sets.

Stable theories

Definition

A formula $\varphi(x, y)$ is *stable* if it does not have the order property, i.e. it does not define a total order on arbitrarily large sets.
(Formally, it means that there is some $n \in \mathbf{N}$ such that there are no $a_1, \ldots, a_n, b_1, \ldots, b_n$ such that $\varphi(a_i, b_j)$ holds if and only if $i \leq j$.)
A (complete) theory is stable if every formula is stable.

Proposition (Grothendieck, Ben Yaacov)

A countable theory T is stable if and only if for every countable $M \models T$, the dynamical systems of the form $(\mathrm{Aut}(M), S_x(M))$ are WAP.

WAP and stability

Proposition (Grothendieck, Ben Yaacov)

A countable theory T is stable if and only if for every countable $M \models T$, the dynamical systems of the form $(\mathrm{Aut}(M), S_x(M))$ are WAP.

Sketch of sketch of proof.

By the WAP testing fact, we can work directly with formulas in both directions. If $\varphi(x, y)$ has the order property, we can find

- a countable model $M \models T$,
- sequences $(a_n)_n, (b_m)_m$ such that $M \models \varphi(a_n, b_m)$ iff $n \leq m$,
- $\sigma_m \in \mathrm{Aut}(M)$ such that $\sigma_m \cdot b_0 = b_m$.

This implies that WAP fails in $(\mathrm{Aut}(M), S_x(M))$. The other direction is similar. □

Tame dynamical systems

Definition

We say that a metric dynamical system (G, X) is *tame* if for every $f \in C(X)$, the pointwise closure of $G \cdot f \subseteq \mathbf{R}^X$ consists of Borel measurable functions (equivalently, $|\overline{G \cdot f}| \leq 2^{\aleph_0}$).

Example

The shift action of $G = \mathbf{Z}$ on $X = 2^{\mathbf{Z}}$ is not tame. To show that, it is enough to notice that for any $\mathcal{U} \in \beta\mathbf{Z}$ and $A \subseteq \mathbf{Z}$, $(\lim_{k \to \mathcal{U}}(-k) \cdot \chi_A)(0) = \lim_{k \to \mathcal{U}} \chi_A(k) = 1$ if and only if $A \in \mathcal{U}$.

Fact (tameness testing)

If X is totally disconnected, it is enough to check tameness on characteristic functions of basic clopen sets.

NIP theories

Definition

We say that a formula $\varphi(x, y)$ has NIP if it does not have the independence property, i.e. it does not define arbitrarily large independent families.
(Formally, this means that there is some k such that for any $a_1, \ldots, a_k$ there is $A \subseteq \{1, \ldots, k\}$ such that for no b we have that $\varphi(a_i, b)$ holds if and only if $i \in A$.)
A (complete) theory has NIP if every formula has NIP.

Proposition (Glasner and Megrelishvili, Ibarlucía)

A countable theory T is NIP if and only if for every countable $M \models T$, the dynamical systems of the form $(\mathrm{Aut}(M), S_x(M))$ are tame.

NIP and tameness

Proposition (Glasner and Megrelishvili, Ibarlucía)

A countable theory T is NIP if and only if for every countable $M \models T$, the dynamical systems of the form $(\mathrm{Aut}(M), S_x(M))$ are tame.

(Sketch of)^3 proof.

The approach is similar to the stable/WAP case: given formula $\varphi(x, y)$ with IP, we find a characteristic function in some $S_x(M)$ with large orbit closure. The other direction is more complicated, and uses the machinery of Bourgain, Fremlin and Talagrand. □

Metamodeltheoretical remarks

- Stability and NIP are crucial notions in model theory,
- essentially equivalent notions appear in different parts of mathematics,
- there are also further ties, e.g. to topology, functional analysis (via different characterisations of WAP/tameness in terms of Rosenthal compacta, ℓ^1-sequences etc.),
- similar connections have been observed in different contexts in model theory, by several people (e.g. Ben Yaacov, Chernikov, Ibarlucía, Pillay, Simon),
- it seems that the fundamental idea underlying those things has not been fully realised so far.

Borel cardinality

Definition

If E, F are equivalence relations on Polish spaces X, Y, then we say that $E \leq_B F$ if there is a Borel $f\colon X \to Y$ such that $x_1 \mathrel{E} x_2 \iff f(x_1) \mathrel{F} x_2$, and we say that $E \sim_B F$ if $E \leq_B F$ and $F \leq_B E$.

Intuitively, Borel cardinality measures difficulty of a classification problem.

Example

If $\Delta(n)$ is the diagonal (equality) on the n-element discrete space, then $\Delta(n) \leq_B \Delta(m)$ if and only if $n \leq m$.

Borel cardinality

Definition

If E, F are equivalence relations on Polish spaces X, Y, then we say that $E \leq_B F$ if there is a Borel $f\colon X \to Y$ such that $x_1 \; E \; x_2 \iff f(x_1) \; F \; x_2$, and we say that $E \sim_B F$ if $E \leq_B F$ and $F \leq_B E$.

Intuitively, Borel cardinality measures difficulty of a classification problem.

Example

$\mathbf{E}_0$ is the equivalence relation on 2^ω given by $f_1 \; \mathbf{E}_0 \; f_2$ iff f_1 and f_2 are eventually equal. We have $\Delta(2^\omega) \leq_B \mathbf{E}_0 \not\leq_B \Delta(2^\omega)$.

Galois group

Definition

If $\mathfrak{C} \models T$ is a monster model (e.g. a large saturated model), then the Galois group $\mathrm{Gal}(T)$ is the quotient $\mathrm{Aut}(\mathfrak{C})/\operatorname{Autf}_L(\mathfrak{C})$, where $\operatorname{Autf}_L(\mathfrak{C})$ is the subgroup of $\mathrm{Aut}(\mathfrak{C})$ generated by all $\mathrm{Aut}(\mathfrak{C}/M)$ for $M \preceq \mathfrak{C}$.

Fact

$\mathrm{Gal}(T)$ *is a compact (possibly non-Hausdorff) topological group, and it has a well-defined Borel cardinality (all independent of* $\mathfrak{C}$*).*

(Both the topology and the Borel cardinality of $\mathrm{Gal}(T)$ can be obtained via a surjection from a space of complete types over a model.)

Polish extension of the Galois group

Theorem (Rz., Krupiński)

If T is a countable theory, there is a compact Polish group $\hat{G}$ and a group epimorphism $\hat{r}\colon \hat{G} \to \mathrm{Gal}(T)$ with the following properties:

1. *$H = \ker \hat{r}$ is an F_σ subgroup of $\hat{G}$,*
2. *$\hat{r}$ is a topological quotient mapping (i.e. $\mathrm{Gal}(T)$ is topologically isomorphic to $\hat{G}/H$),*
3. *$\hat{G}/H \leq_B \mathrm{Gal}(T)$, and if T has NIP, then $\hat{G}/H \sim_B \mathrm{Gal}(T)$.*

The same idea applies to certain strong type spaces and quotients of type-definable groups by connected group components.

Canonical (?) Polish extension of the Galois group

- The proof of the theorem relies heavily on topological dynamics; in particular, we use the derived subgroup $u\mathcal{M}/H(u\mathcal{M})$ of the Ellis group of a dynamical system of the form $(\mathrm{Aut}(M), S_m(M))$ for a suitably chosen countable model M.
- In the NIP case, we use the Bourgain-Fremlin-Talagrand dichotomy (similarly to preceding proposition).
- A priori, the resulting $\hat{G}$ (and $\hat{r}$) depends heavily on the choice of M.

Conjecture

If T is a countable NIP theory, then there is are canonical $\hat{r}$, $\hat{G}$ as in the theorem.