When do closed classes imply closedness?
(and more)

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General setup

- G is a group,
- X is a set on which G acts,
- E is a G-invariant equivalence relation on X
(i.e. $x\,E\,y \iff gx\,E\,gy$, for all x, y, g).
- [Usually, we want to have $x\,E\,y \implies G\cdot x = G\cdot y$.]

Possible constraints:

- The action is continuous.
- G is compact.
- (G, X are type-definable.)
- Etc.

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Problem

Question

If G is a topological group acting continuously on X, under what conditions is it true that E has closed classes iff it is closed [iff X/E is somehow nice]?

Example

- Consider $G = \mathbf{R}$ acting on $X = \mathbf{R}^2$ by $r \cdot (a, b) = (a + rb, b)$, and let E be the orbit equivalence relation.
- Classes of E are points on the line $y = 0$ and lines parallel to it.
- The classes of E are closed, and the class space is easy to understand (it is almost Hausdorff), but E is not closed.

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Very simple remark

Remark

Suppose G is a topological group and $H \leq G$. Then H is closed iff the relation E_H of lying in the same left coset of H is closed.

Proof.

$E_H = \{(g_1, g_2) \mid g_1^{-1}g_2 \in H\}$, while $H = \{g \in G \mid g\, E_H\, e\}$, so E_H is the preimage of H by the (continuous) map $(g_1, g_2) \mapsto g_1^{-1}g_2$ and H is the section of E_H at e. □

Remark

If E is invariant on $X = G$, then $E = E_H$, where $H = [e]_E$.

Example

We want at the very least to have that $G\backslash X$ is nice. For arbitrary continuous actions, various pathologies are possible.

Example

Consider $G = \mathbf{R}$ acting on the flat torus $\mathbf{R}^2/\mathbf{Z}^2$ by $r \cdot (a, b) = (a + r, b + r\alpha)$ for a fixed $\alpha \notin \mathbf{Q}$, and let E be the orbit equivalence relation.
Then E does not have closed classes (they are all meagre and dense), and the quotient space has no nice structure (e.g. the topology is trivial).

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Suppose G, X are compact, G acts transitively and continuously on X and E is invariant.
Then if E has closed classes, it is closed (in X^2).

Proof.

- Fix $x_0 \in X$, put $H := \mathrm{Stab}_G[x_0]_E$ (preimage of $[x_0]_E$ via the orbit map $\varphi_{x_0} : g \mapsto g \cdot x_0$).
- Then $g_1 x_0 \mathrel{E} g_2 x_0$ iff $g_1 H = g_2 H$.
- It follows that $E = \{(g_1 x_0, g_2 x_0) \mid g_1 \mathrel{E_H} g_2\}$, i.e. E is the pushforward of E_H via φ_{x_0}, so it is closed by compactness. □

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Gluing orbits

Remark

- If E is finer than the orbit equivalence relation of G, then E is invariant iff each $E\restriction_{G\cdot x}$ is invariant.
- If G-orbits are closed and E is as above, then E has closed classes iff each $E\restriction_{G\cdot x}$ has closed classes.
- In general, there is no reason for E to be closed even if all $E\restriction_{G\cdot x}$ are closed.

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Simple counterexample

Example

- Let $G = \mathbf{Z}/2\mathbf{Z}$ act on $X = G \times \{0, 1/n \mid n \in \mathbf{N}\}$ in the natural way.
- Put $(g, x)\ E\ (h, y)$ iff $x = y$ and $(g = h$ or $x \neq 0)$.
- E is invariant, its classes are closed, but it is not closed.

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This shows that we cannot just extend the preceding remark to intransitive actions.
We need to impose some additional condition on consistency of E across G-orbits.

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Smoothness

Suppose we have an equivalence relation E on a Polish (=separable completely metrisable) space X. We say that E or X/E is smooth ("classifiable by reals") if we can attach, in a Borel way, a unique real number to each E-class.

Fact

Any G_δ (in particular, any closed) equivalence relation is smooth.

Example

If we consider the action of $\mathbf{Z}$ on the circle S^1 by an irrational rotation, then $S^1/\mathbf{Z}$ is not smooth.

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Proof.

Suppose towards contradiction that $S^1/\mathbf{Z}$ is smooth, and let $f\colon S^1 \to \mathbf{R}$ be the Borel function witnessing that.
Then f is Baire measurable, so there is a comeagre set $C \subseteq S^1$ such that $f\restriction_C$ is continuous.
Therefore, $C' := \bigcap_{z\in\mathbf{Z}} z \cdot C$ is also comeagre and $\mathbf{Z}$-invariant, so it contains more than one orbit (because orbits are meagre).
But orbits are dense and f is constant on each of them, which is a contradiction, as f is continuous on C'. □

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Miller's theorem

Fact (Miller, 1977)

Suppose G is a Polish group acting on $X = G$ and $H \leq G$. If E_H is smooth, then H is closed.

Corollary (solution for the "transitive" case)

If G is a compact Polish group, acting continuously and transitively on a Polish space X, TFAE for invariant E on X:

1. *E is closed,*
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Smoothness and closedness

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Example

Suppose E is a (nontrivial) rotation-invariant equivalence relation on S^1. Then E is smooth exactly when $E = E_\theta$, where $z_1 \; E_\theta \; z_2 \iff z_1/z_2 = e^{k\theta i}$ for some integer k, where θ is a (fixed) rational multiple of π.

Idea

- In the transitive case, we had $E = \varphi_{x_0}[E_H]$.
- In general, given E refining G-orbit equivalence, we have $E = \bigcup_{x \in X} E\restriction_{G \cdot x}$.
- If each $G \cdot x$ is closed, and if E is smooth, then so are all $E\restriction_{G \cdot x}$.
- Then for compact [Polish] G, X we have:

$$E \text{ closed } \implies [E \text{ smooth } \implies E\restriction_{G\cdot x} \text{ all smooth } \implies] \implies [x]_E \text{ all closed } \implies E\restriction_{G\cdot x} \text{ all closed}$$

- For the missing implication, we need to have a better description of E than $E = \bigcup_{x \in X} E\restriction_{G \cdot x}$.
- Intuitively, we want to have that the transition between "stalks" $E\restriction_{G \cdot x}$ is "continuous".

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Orbital equivalence relations

Definition

(An invariant equivalence relation) E is orbital if there is some $H \leq G$ such that for all x, $[x]_E = H \cdot x$.

Intuition: E is orbital if for some H, the relation E is the "uniform pushforward" of E_H by all the orbit maps; note that:

$$E = \bigcup_{x \in X} E\restriction_{G \cdot x} = \bigcup_{x \in X} \varphi_x[E_H]$$

Example

Suppose $G = X$ and $H \leq G$. Then E_H is orbital iff H is normal. (Because an orbital equivalence relation on G is the relation of lying in the right coset of a subgroup of G, and E_H is the relation of lying in the same left coset.)

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Lemma

Suppose E is orbital. Then orbitality is witnessed by $H = \bigcap_{x \in X} \mathrm{Stab}_G [x]_E$.

Proof.

Let H' witness orbitality of E. Then for all x we have $H' \cdot x = [x]_E \subseteq [x]_E$ – so $H' \leq H$. On the other hand, we have by definition $H \cdot x \subseteq [x]_E = H' \cdot x$ – so $H \cdot x \subseteq [x]_E \subseteq H \cdot x$. □

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Proposition

Suppose G and X are compact and E is orbital. Then E is closed iff all of its classes are closed.

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Let H witness orbitality of E; then $E = \{(x, hx) \mid h \in H, x \in X\}$, which is closed if H is closed in G. But $\bigcap_{x \in X} \mathrm{Stab}_G[x]_E$ is closed and it witnesses orbitality. □

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Smoothness for orbital equivalence relations

Proposition

Suppose G and X are compact and E is orbital. Then E is closed iff all of its classes are closed.

Corollary (solution for the "orbital" case)

If G and X are compact Polish, while E is orbital, TFAE:

1. *E is closed,*
2. *E has closed classes,*
3. *E is smooth.*

Proof.

If E is smooth, so are all the $E\restriction_{G\cdot x}$, the classes are closed by the transitive case. □

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Corollary (solution for the "orbital" case)

If G and X are compact Polish, while E is orbital, TFAE:

1. *E is closed,*
2. *E has closed classes,*
3. *E is smooth.*

Example

Consider $G = \mathrm{SO}(2)$ acting on $X = D^2$. The smooth orbital equivalence relations are exactly those induced by finite subgroups of G.

Weakly orbital equivalence relations

- The "transitive" and "orbital" cases can be generalised together in the form of "weakly orbital" equivalence relations.
- They can be intuitively interpreted as the relations which are "non-uniform" but still somehow consistent pushforwards of some E_H via the orbit maps.
- Naively: put $E = \bigcup_{x \in X} \varphi_x[E_{H_x}]$ for some $H_x \leq G$. But this is trivial: just take $H_x = \mathrm{Stab}_G[x]_E$! We need more control over H_x.

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General (?) solution to the problem

Proposition (solution for the "weakly orbital" case)

Suppose G, X are compact Polish, and E is weakly orbital*. The following are equivalent:

1. *E is closed,*
2. *E has closed classes,*
3. *E is smooth.*

(The * denotes an extra assumption of "continuity" of the "non-uniform" pushforward.)

Invariant equivalence relations in model theory

We consider a monster model $\mathfrak{C}$ of a fixed complete first order theory T (i.e. a structure satisfying certain axioms and very rich in automorphisms).

Definition

We say that a set in (a power of) $\mathfrak{C}$ is *invariant* when it is $\mathrm{Aut}(\mathfrak{C})$-invariant.

Definition

An invariant equivalence relation on some $X \subseteq \mathfrak{C}$ is bounded if $|X/E| \leq 2^{|T|}$ (BIER). (Intuitively, X/E does not depend on $\mathfrak{C}$, only T.)

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Lascar equivalence, (type-)definability

Definition

On any invariant set X, there is a finest BIER on X, denoted by $\equiv_L$.

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We say that a set $X \subseteq \mathfrak{C}$ is definable or "pseudo-clopen" (with parameters $a_1, \ldots, a_n$) if there is a formula $\varphi(x, a_1, \ldots, a_n)$ in the language of T such that $X = \{b \in \mathfrak{C} \mid \varphi(b, a_1, \ldots, a_n) \text{ is true}\}$

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We call a set [$\emptyset$-]type-definable or "pseudo-closed" when it is the intersection of a small family of sets definable [without parameters].

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Smoothness in model theory

The notion of smoothness has been adapted to model theory in the following form.

Definition

Suppose E is a BIER and T is countable. Then E^M is the "pushforward" of E to the space of types over a countable model M (=orbits of $\mathrm{Aut}(\mathfrak{C}/M)$).

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Previous results

Fact (Newelski 2002)

If $\equiv_L$ is not type-definable on X, then $|X/\equiv_L| \geq 2^{\aleph_0}$

Conjecture (Krupiński, Pillay, Solecki 2012)

If T is countable, then $\equiv_L$ on a $\emptyset$-type-definable set is smooth iff it is type-definable ("closed").

(Type-definable $\Rightarrow$ smooth is easy.)

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Fact (Krupiński, Pillay, Rz. 2015)

If E is a BIER on some $[\alpha]_\equiv$ (i.e. an $\mathrm{Aut}(\mathfrak{C})$-orbit) and T is countable, then E is smooth iff it is type-definable.

Remark

- This is not true if the domain of E is not a complete type (i.e. $\mathrm{Aut}(\mathfrak{C})$ does not act transitively on it).
- The reason is the same as in the topological case: essentially, gluing together countably many pieces preserves smoothness, but not type-definability.

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- The fact does hold more generally, for example, in case of $\equiv_L$ on any $\emptyset$-type-definable set.
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Theorem

Suppose E is a BIER (on a $\emptyset$-type-definable set), weakly orbital with respect to $G = \mathrm{Aut}(\mathfrak{C})$, while T is countable. Then TFAE:*

- *E is type-definable,*
- *all E-classes are type-definable,*
- *E is smooth.*

Idea of the proof.

Such E can be seen as a "regular pushforward" of an equivalence relation on $\mathrm{Gal}(T)$, which allows us to reduce to the case of transitive $\mathrm{Aut}(\mathfrak{C})$-actions, which is the content of the preceding fact. □

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(Type-)definable group actions

If G and X are $\emptyset$-type-definable, as is the group action, while all E-classes are (setwise) $G^{000}_{\emptyset}$-invariant, we can prove a similar result by essentially the same methods.
(Note: $G^{000}_{\emptyset}$ is the so-called connected component of G.)

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Suppose G, X, E are as above, and E is weakly orbital. Then E is type-definable iff all its classes are type-definable.*

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Definable group actions and smoothness

Fact (Krupiński, Pillay, Rz. 2015)

Suppose G is $\emptyset$-definable, T is countable, and $H \leq G$ is invariant of bounded index. Then E_H is smooth iff H is type-definable.

(Note that H is invariant of bounded index iff E_H is a BIER.)

Corollary

Suppose that T is countable, G and X are $\emptyset$-definable, as is the action. If E is a weakly orbital*, G-invariant BIER on X, then E is smooth iff it is type-definable.

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