Smoothness of bounded invariant equivalence relations

Tomasz Rzepecki
(joint work with Krzysztof Krupiński)

Uniwersytet Wrocławski

Classification Theory Workshop, 2014

Motivations

- General goal: understanding "strong type spaces".
- A theorem of Newelski (2002) about F_σ equivalence relations (cardinality).
- A theorem of Kaplan, Miller and Simon (2013) about Borel cardinality of Lascar strong type (Borel cardinality).
- A question of Gismatullin and Krupiński (2012) related to connected group components.

Motivations

- General goal: understanding "strong type spaces".
- A theorem of Newelski (2002) about F_σ equivalence relations (cardinality).
- A theorem of Kaplan, Miller and Simon (2013) about Borel cardinality of Lascar strong type (Borel cardinality).
- A question of Gismatullin and Krupiński (2012) related to connected group components.

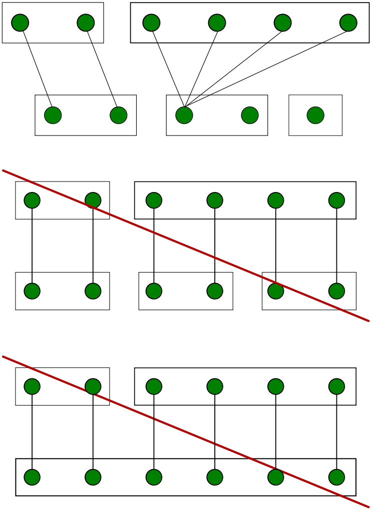
Motivations

- General goal: understanding “strong type spaces”.
- A theorem of Newelski (2002) about F_σ equivalence relations (cardinality).
- A theorem of Kaplan, Miller and Simon (2013) about Borel cardinality of Lascar strong type (Borel cardinality).
- A question of Gismatullin and Krupiński (2012) related to connected group components.

Motivations

- General goal: understanding “strong type spaces”.
- A theorem of Newelski (2002) about F_σ equivalence relations (cardinality).
- A theorem of Kaplan, Miller and Simon (2013) about Borel cardinality of Lascar strong type (Borel cardinality).
- A question of Gismatullin and Krupiński (2012) related to connected group components.

Borel reductions

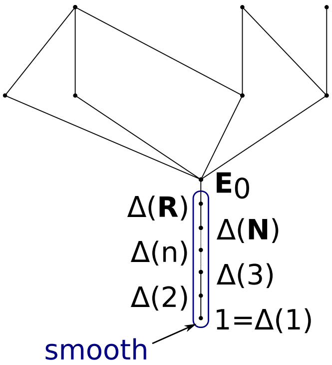


Definition

Suppose X, Y are Polish spaces and E, F are Borel equivalence relations on X, Y. Then $f\colon X \to Y$ is a Borel reduction of E to F if

$$f(x)\ F\ f(x') \iff x\ E\ x'$$

Borel cardinalities



Definition

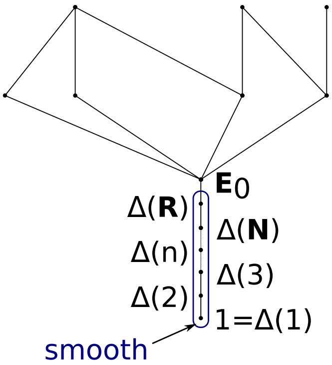
$E \leq_B F$ if there exists a Borel reduction of E to F.

$E \sim_B F$ if $E \leq_B F$ and $F \leq_B E$; E is smooth if $E \sim_B \Delta(X)$.

Fact

- *There is a smallest non-smooth equivalence relation,* $\mathbf{E}_0$.
- $\leq_B$ *is linear up to* $\mathbf{E}_0$.

Borel cardinalities



Definition

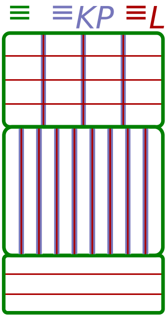
$E \leq_B F$ if there exists a Borel reduction of E to F.

$E \sim_B F$ if $E \leq_B F$ and $F \leq_B E$; E is smooth if $E \sim_B \Delta(X)$.

Fact

- *There is a smallest non-smooth equivalence relation,* $\mathbf{E}_0$*.*
- $\leq_B$ *is linear up to* $\mathbf{E}_0$*.*

Strong types



Definition

$\equiv_{KP}$ is the finest bounded (i.e. with small number of classes), $\emptyset$-type-definable equivalence relation.

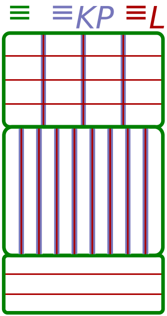
$\equiv_L$ is the finest bounded, invariant equivalence relation.

Fact

$\equiv_L$ *is* $(\emptyset-)F_\sigma$*, i.e.* $x \equiv_L y \iff \bigvee_n \Phi_n(x, y)$.

In the sequel, E is a bounded, F_σ equivalence relation on a $\emptyset$-type-definable set $X \subseteq \mathfrak{C}$

Strong types



Definition

$\equiv_{KP}$ is the finest bounded (i.e. with small number of classes), $\emptyset$-type-definable equivalence relation.

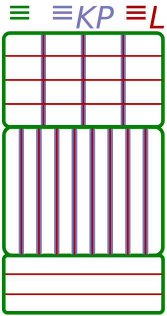
$\equiv_L$ is the finest bounded, invariant equivalence relation.

Fact

$\equiv_L$ is $(\emptyset-)F_\sigma$, i.e. $x \equiv_L y \iff \bigvee_n \Phi_n(x,y)$.

In the sequel, E is a bounded, F_σ equivalence relation on a $\emptyset$-type-definable set $X \subseteq \mathfrak{C}$

Strong types



Definition

$\equiv_{KP}$ is the finest bounded (i.e. with small number of classes), $\emptyset$-type-definable equivalence relation.

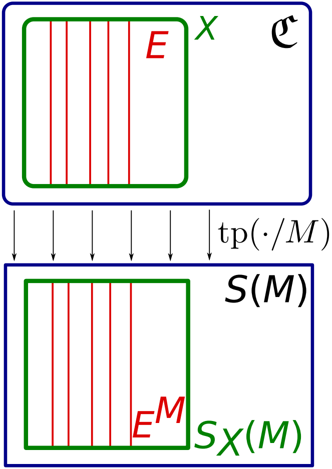
$\equiv_L$ is the finest bounded, invariant equivalence relation.

Fact

$\equiv_L$ is ($\emptyset$−)F_σ, i.e. $x \equiv_L y \iff \bigvee_n \Phi_n(x, y)$.

In the sequel, E is a bounded, F_σ equivalence relation on a $\emptyset$-type-definable set $X \subseteq \mathfrak{C}$

Borel cardinalities of invariant equivalence relations



Definition

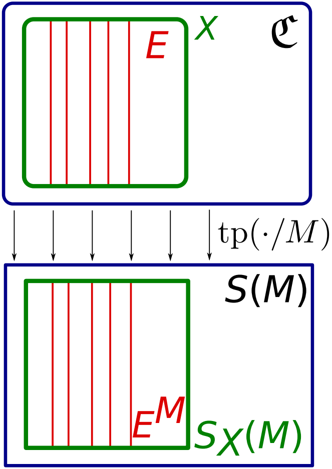
Borel cardinality of E is the Borel cardinality of E^M for a ctble model M.

$$p\, E^M\, q \iff \exists a \models p\, \exists b \models q\, (a\, E\, b)$$

Remark

Type-definable ERs (e.g. $\equiv_{KP}$) are smooth.

Borel cardinalities of invariant equivalence relations



Definition

Borel cardinality of E is the Borel cardinality of E^M for a ctble model M.

$$p\, E^M\, q \iff \exists a \models p\, \exists b \models q\, (a\, E\, b)$$

Remark

Type-definable ERs (e.g. $\equiv_{KP}$) are smooth.

Orbital equivalence relations

Definition

As before, E is an F_σ, bounded equivalence relation on $X \subseteq \mathfrak{C}$.

E is orbital if there is some $\Gamma \leq \mathrm{Aut}(\mathfrak{C})$ such that E-classes are orbits of Γ.

E is orbital on types if it refines $\equiv$ and restrictions of E to types are orbital.

Example

$\equiv_{KP}$ and $\equiv_L$ are orbital.

Orbital equivalence relations

Definition

As before, E is an F_σ, bounded equivalence relation on $X \subseteq \mathfrak{C}$.

E is orbital if there is some $\Gamma \leq \mathrm{Aut}(\mathfrak{C})$ such that E-classes are orbits of Γ.

E is orbital on types if it refines $\equiv$ and restrictions of E to types are orbital.

Example

$\equiv_{KP}$ and $\equiv_L$ are orbital.

Orbital on types vs refining $\equiv$

Question

If E refines $\equiv$, is E already orbital on types?

Answer

No! (We have found counterexamples.)

Normal form

Definition

If E is F_σ on X, then $\bigvee_n \Phi_n(x,y)$ is a normal form for E if

- it is increasing (i.e. $\Phi_n \vdash \Phi_{n+1}$),
- $E(x,y) \iff \bigvee_n \Phi_n(x,y)$,
- $d(x,y) = \min\{n \mid \mathfrak{C} \models \Phi_n(x,y)\}$ is a metric on X.

Example

$\equiv_L$ is has normal form $\Phi_n(x,y) =$ "$d_L(x,y) \leq n$" (Lascar distance).

Normal form

Definition

If E is F_σ on X, then $\bigvee_n \Phi_n(x,y)$ is a normal form for E if

- it is increasing (i.e. $\Phi_n \vdash \Phi_{n+1}$),
- $E(x,y) \iff \bigvee_n \Phi_n(x,y)$,
- $d(x,y) = \min\{n \mid \mathfrak{C} \models \Phi_n(x,y)\}$ is a metric on X.

Example

$\equiv_L$ is has normal form $\Phi_n(x,y) =$ "$d_L(x,y) \leq n$" (Lascar distance).

Normal form

Definition

If E is F_σ on X, then $\bigvee_n \Phi_n(x,y)$ is a normal form for E if

- it is increasing (i.e. $\Phi_n \vdash \Phi_{n+1}$),
- $E(x,y) \iff \bigvee_n \Phi_n(x,y)$,
- $d(x,y) = \min\{n \mid \mathfrak{C} \models \Phi_n(x,y)\}$ is a metric on X.

Example

$\equiv_L$ is has normal form $\Phi_n(x,y) =$ "$d_L(x,y) \leq n$" (Lascar distance).

Normal form

Definition

If E is F_σ on X, then $\bigvee_n \Phi_n(x,y)$ is a normal form for E if

- it is increasing (i.e. $\Phi_n \vdash \Phi_{n+1}$),
- $E(x,y) \iff \bigvee_n \Phi_n(x,y)$,
- $d(x,y) = \min\{n \mid \mathfrak{C} \models \Phi_n(x,y)\}$ is a metric on X.

Example

$\equiv_L$ is has normal form $\Phi_n(x,y) =$ "$d_L(x,y) \leq n$" (Lascar distance).

Normal forms exist

Proposition

If E is bounded F_σ, then E has a normal form $\bigvee_n \Phi_n$ such that "$d_L(x,y) \leq n$" $\vdash \Phi_n(x,y)$.

Proposition

C: an E-class; assume that E refines $\equiv$. TFAE:

- *C has infinite diameter w.r.t. some normal form;*
- *C has infinite diameter w.r.t. every normal form.*

Proof.

Easy from a theorem of Newelski. □

Normal forms exist

Proposition

If E is bounded F_σ, then E has a normal form $\bigvee_n \Phi_n$ such that "$d_L(x,y) \leq n$" $\vdash \Phi_n(x,y)$.

Proposition

C: an E-class; assume that E refines $\equiv$. TFAE:

- *C has infinite diameter w.r.t. some normal form;*
- *C has infinite diameter w.r.t. every normal form.*

Proof.

Easy from a theorem of Newelski. □

Normal forms exist

Proposition

If E is bounded F_σ, then E has a normal form $\bigvee_n \Phi_n$ such that "$d_L(x,y) \leq n$" $\vdash \Phi_n(x,y)$.

Proposition

C: an E-class; assume that E refines $\equiv$. TFAE:

- *C has infinite diameter w.r.t. some normal form;*
- *C has infinite diameter w.r.t. every normal form.*

Proof.

Easy from a theorem of Newelski. □

Technical theorem

Fact (Newelski 2002, simplified)

E: F_σ, refines $\equiv$.
Then if $[a]_E$ has infinite diameter, then $E\restriction_{[a]_\equiv}$ has at least $\mathfrak{c}$ classes.

Fact (Kaplan, Miller & Simon 2013)

The class $[a]_{\equiv_L}$ has infinite diameter (w.r.t. Lascar distance) iff $\equiv_L\restriction_{[a]_\equiv}$ is non-smooth.

Theorem (countable case; independently Kaplan & Miller 2013)

E: is bdd, F_σ and orbital on types.
Then $[a]_E$ has infinite diameter $\iff$ $E\restriction_{[a]_\equiv}$ is non-smooth.

Technical theorem

Fact (Newelski 2002, simplified)

E: F_σ, refines $\equiv$.
Then if $[a]_E$ has infinite diameter, then $E{\restriction}_{[a]_\equiv}$ has at least $\mathfrak{c}$ classes.

Fact (Kaplan, Miller & Simon 2013)

The class $[a]_{\equiv_L}$ has infinite diameter (w.r.t. Lascar distance) iff $\equiv_L{\restriction}_{[a]_\equiv}$ is non-smooth.

Theorem (countable case; independently Kaplan & Miller 2013)

E: is bdd, F_σ and orbital on types.
Then $[a]_E$ has infinite diameter $\iff$ $E{\restriction}_{[a]_\equiv}$ is non-smooth.

Technical theorem

Fact (Newelski 2002, simplified)

E: F_σ, refines $\equiv$.
Then if $[a]_E$ has infinite diameter, then $E\restriction_{[a]_\equiv}$ has at least $\mathfrak{c}$ classes.

Fact (Kaplan, Miller & Simon 2013)

The class $[a]_{\equiv_L}$ has infinite diameter (w.r.t. Lascar distance) iff $\equiv_L\restriction_{[a]_\equiv}$ is non-smooth.

Theorem (countable case; independently Kaplan & Miller 2013)

E: is bdd, F_σ and orbital on types.
Then $[a]_E$ has infinite diameter $\iff$ $E\restriction_{[a]_\equiv}$ is non-smooth.

Technical theorem cont.

Corollary (countable case for groups)

G: definable and $N \trianglelefteq G$: F_σ of bounded index. Then N is $\emptyset$-type-definable $\iff$ E_N is smooth.

Remark

Similar result in uncountable case, more complicated techniques.

Technical theorem cont.

Corollary (countable case for groups)

G: definable and $N \trianglelefteq G$: F_σ of bounded index.
Then N is $\emptyset$-type-definable $\iff$ E_N is smooth.

Remark

Similar result in uncountable case, more complicated techniques.

Five conditions theorem

Theorem

E: orbital on types, F_σ, bdd. ER on X:

$$\Uparrow \text{ if } X = p(\mathfrak{C}) \left\{ \begin{array}{c} \textit{Bounded diameters} \\ \Downarrow \\ \textit{Type-definable} \\ \Downarrow \\ \textit{Smooth} \Leftrightarrow \textit{finite diameters} \\ \Downarrow \\ E \textit{ is refined by } \equiv_{KP} \end{array} \right. \Uparrow \text{ if } E \subseteq \equiv_{KP}$$

Five conditions theorem

Theorem

E: orbital on types, F_σ, bdd. ER on X:

$$\Uparrow \text{ if } X = p(\mathfrak{C}) \left\{ \begin{array}{c} \textit{Bounded diameters} \\ \Downarrow \\ \textit{Type-definable} \\ \Downarrow \\ \textit{Smooth} \Leftrightarrow \textit{finite diameters} \end{array} \right.$$

$$\left. \begin{array}{c} \textit{Type-definable} \\ \Downarrow \\ \textit{Smooth} \Leftrightarrow \textit{finite diameters} \\ \Downarrow \\ E \textit{ is refined by } \equiv_{KP} \end{array} \right\} \Uparrow \text{ if } E \subseteq \equiv_{KP}$$

Five conditions theorem

Theorem

E: orbital on types, F_σ, bdd. ER on X:

$$\Uparrow \text{ if } X = p(\mathfrak{C}) \left\{ \begin{array}{c} \text{Bounded diameters} \\ \Downarrow \\ \text{Type-definable} \\ \Downarrow \\ \text{Smooth} \Leftrightarrow \text{finite diameters} \\ \Downarrow \\ E \text{ is refined by } \equiv_{KP} \end{array} \right\} \Uparrow \text{ if } E \subseteq \equiv_{KP}$$

Additional comments

Proposition

Reverse ($\Uparrow$) implications do not hold.

Proof.

Series of counterexamples. □

Question

Can we weaken "orbital on types" assumption?

Answer (partial)

Not too much: E must at least refine $\equiv$.

Additional comments

Proposition

Reverse ($\Uparrow$) implications do not hold.

Proof.

Series of counterexamples. □

Question

Can we weaken “orbital on types” assumption?

Answer (partial)

Not too much: E must at least refine $\equiv$.

Additional comments

Proposition

Reverse ($\Uparrow$) implications do not hold.

Proof.

Series of counterexamples. □

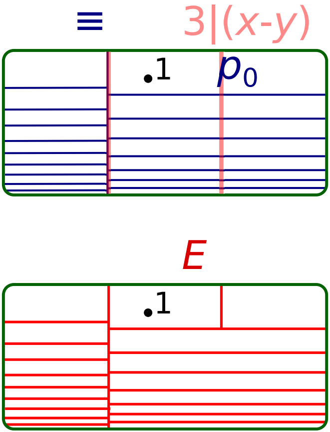
Question

Can we weaken "orbital on types" assumption?

Answer (partial)

Not too much: E must at least refine $\equiv$.

Examples



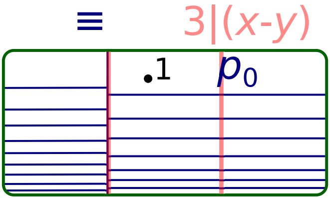
Example

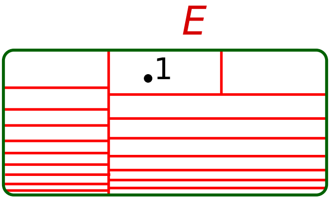
- $T = \mathrm{Th}(\mathbf{Z}, +, n|\cdot)_{n\in\mathbf{N}}$;
- $p_0(x) = \mathrm{tp}(1/\emptyset) = \bigwedge_n n\nmid x$
- $x\, E\, y \iff x \equiv y$ and if $x \models p_0$, then $3|(x-y)$.

E:

- is smooth, but not type-definable,
- is orbital on types (so theorem applies).

Examples





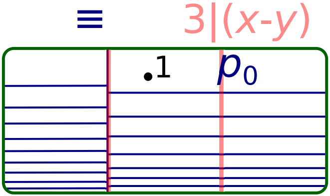
Example

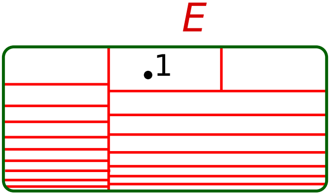
- $T = \mathrm{Th}(\mathbf{Z}, +, n|\cdot)_{n\in\mathbf{N}}$;
- $p_0(x) = \mathrm{tp}(1/\emptyset) = \bigwedge_n n \nmid x$
- $x\,E\,y \iff x \equiv y$ and if $x \models p_0$, then $3|(x-y)$.

E:

- is smooth, but not type-definable,
- is orbital on types (so theorem applies).

Examples





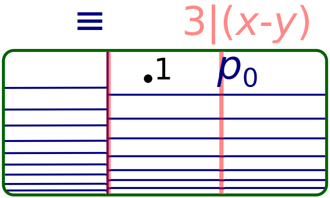
Example

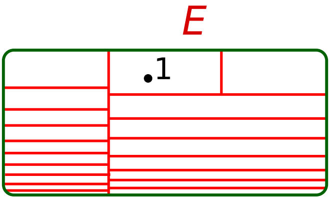
- $T = \mathrm{Th}(\mathbf{Z}, +, n|\cdot)_{n\in\mathbf{N}}$;
- $p_0(x) = \mathrm{tp}(1/\emptyset) = \bigwedge_n n \nmid x$
- $x\,E\,y \iff x \equiv y$ and if $x \models p_0$, then $3|(x-y)$.

E:

- is smooth, but not type-definable,
- is orbital on types (so theorem applies).

Examples





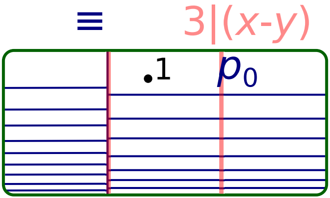
Example

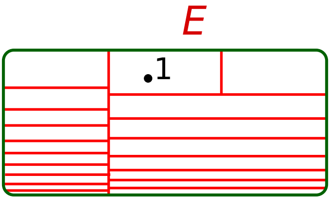
- $T = \mathrm{Th}(\mathbf{Z}, +, n|\cdot)_{n\in\mathbf{N}}$;
- $p_0(x) = \mathrm{tp}(1/\emptyset) = \bigwedge_n n \nmid x$
- $x\,E\,y \iff x \equiv y$ and if $x \models p_0$, then $3|(x-y)$.

E:

- is smooth, but not type-definable,
- is orbital on types (so theorem applies).

Examples





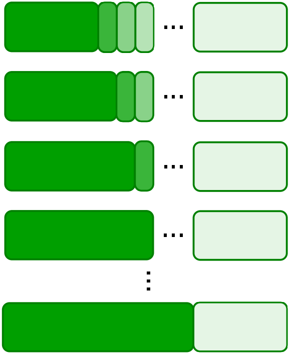
Example

- $T = \mathrm{Th}(\mathbf{Z}, +, n|\cdot)_{n\in\mathbf{N}}$;
- $p_0(x) = \mathrm{tp}(1/\emptyset) = \bigwedge_n n \nmid x$
- $x\ E\ y \iff x \equiv y$ and if $x \models p_0$, then $3|(x-y)$.

E:

- is smooth, but not type-definable,
- is orbital on types (so theorem applies).

Examples

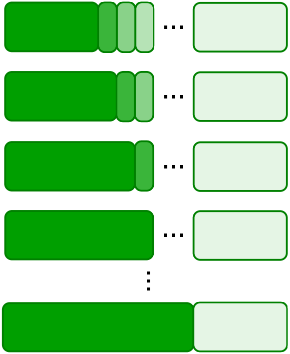


Example

- $T = \mathrm{Th}(\mathbf{R}, +, \cdot, 1, 0, <)$;
- $\Phi_n(x, y) = \bigwedge_{m \geq n}(x < m \leftrightarrow y < m)$;
- $E = \bigvee_n \Phi_n(x, y)$

E has only 2 classes (so it is smooth), although one class has infinite diameter (and E is not type-definable).
E does not refine $\equiv$ (theorem does not apply).

Examples

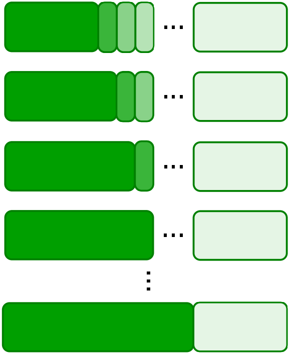


Example

- $T = \mathrm{Th}(\mathbf{R}, +, \cdot, 1, 0, <)$;
- $\Phi_n(x, y) = \bigwedge_{m \geq n}(x < m \leftrightarrow y < m)$;
- $E = \bigvee_n \Phi_n(x, y)$

E has only 2 classes (so it is smooth), although one class has infinite diameter (and E is not type-definable).
E does not refine $\equiv$ (theorem does not apply).

Examples

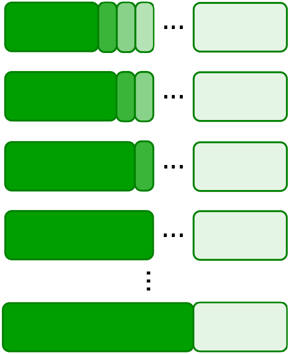


Example

- $T = \mathrm{Th}(\mathbf{R}, +, \cdot, 1, 0, <)$;
- $\Phi_n(x, y) = \bigwedge_{m \geq n}(x < m \leftrightarrow y < m)$;
- $E = \bigvee_n \Phi_n(x, y)$

E has only 2 classes (so it is smooth), although one class has infinite diameter (and E is not type-definable).
E does not refine $\equiv$ (theorem does not apply).

Examples



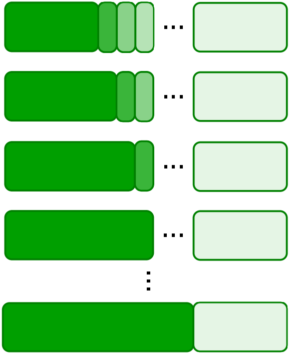
Example

- $T = \mathrm{Th}(\mathbf{R}, +, \cdot, 1, 0, <)$;
- $\Phi_n(x, y) = \bigwedge_{m \geq n}(x < m \leftrightarrow y < m)$;
- $E = \bigvee_n \Phi_n(x, y)$

E has only 2 classes (so it is smooth), although one class has infinite diameter (and E is not type-definable).

E does not refine $\equiv$ (theorem does not apply).

Examples



Example

- $T = \mathrm{Th}(\mathbf{R}, +, \cdot, 1, 0, <)$;
- $\Phi_n(x, y) = \bigwedge_{m \geq n}(x < m \leftrightarrow y < m)$;
- $E = \bigvee_n \Phi_n(x, y)$

E has only 2 classes (so it is smooth), although one class has infinite diameter (and E is not type-definable).
E does not refine $\equiv$ (theorem does not apply).

Connected components

Definition

G: ($\emptyset$-) definable group in $\mathfrak{C}$.

G^{00} : the smallest type-definable subgroup of bounded index;

G^{000} : the smallest invariant subgroup of bounded index.

Question

Are there definable groups such that $G^{00} \neq G^{000}$?

Answer (Pillay & Conversano 2012)

Yes! ($\widetilde{SL_2(\mathbf{R})}^*$)

Connected components

Definition

G: ($\emptyset$-) definable group in $\mathfrak{C}$.

G^{00} : the smallest type-definable subgroup of bounded index;

G^{000} : the smallest invariant subgroup of bounded index.

Question

Are there definable groups such that $G^{00} \neq G^{000}$?

Answer (Pillay & Conversano 2012)

Yes! ($\widetilde{SL_2(\mathbf{R})}^*$)

Connected components

Definition

G: ($\emptyset$-) definable group in $\mathfrak{C}$.

G^{00} : the smallest type-definable subgroup of bounded index;

G^{000} : the smallest invariant subgroup of bounded index.

Question

Are there definable groups such that $G^{00} \neq G^{000}$?

Answer (Pillay & Conversano 2012)

Yes! ($\widetilde{SL_2(\mathbf{R})}^*$)

Connected components: meta-example

Theorem (Gismatullin, Krupiński 2012)

$$0 \to A \to \widetilde{G} \xrightarrow{\pi} G \to 0$$

A, G: *definable groups (A abelian),*

$\widetilde{G} = A \times G$: *definable group in terms of 2-cocycle $h\colon G^2 \to A$.*

Under some technical assumptions and assumption † (concerning non-splitting of a cocycle derived from h), we have

$$\widetilde{G}^{00} \neq \widetilde{G}^{000}.$$

Connected components: meta-example

Theorem (Gismatullin, Krupiński 2012)

$$0 \to A \to \widetilde{G} \xrightarrow{\pi} G \to 0$$

A, G: *definable groups (A abelian),*

$\widetilde{G} = A \times G$: *definable group in terms of 2-cocyle $h\colon G^2 \to A$.*

Under some technical assumptions and assumption †
(concerning non-splitting of a cocycle derived from h), we have

$$\widetilde{G}^{00} \neq \widetilde{G}^{000}.$$

Connected components: meta-example

Theorem (Gismatullin, Krupiński 2012)

$$0 \to A \to \widetilde{G} \xrightarrow{\pi} G \to 0$$

A, G: *definable groups (A abelian),*

$\widetilde{G} = A \times G$: *definable group in terms of 2-cocycle $h\colon G^2 \to A$.*

Under some technical assumptions and assumption † (concerning non-splitting of a cocycle derived from h), we have

$$\widetilde{G}^{00} \neq \widetilde{G}^{000}.$$

Main theorem for definable group extensions

Theorem

Suppose we have:

- $\widetilde{H} \trianglelefteq \widetilde{G}$: F_σ *normal subgroup;*
- $\widetilde{H} \cap A$ *and* $\pi[\widetilde{H}]$*: type-definable;*
- *(technical assumptions).*

Then $\widetilde{H}$ *is type-definable.*

Proof.

Using the technical theorem and a certain topology (weaker than Vietoris) on subsets of $A/(A \cap \widetilde{H})$. □

Main theorem for definable group extensions

Theorem

Suppose we have:

- $\widetilde{H} \trianglelefteq \widetilde{G}$: F_σ *normal subgroup;*
- $\widetilde{H} \cap A$ *and* $\pi[\widetilde{H}]$: *type-definable;*
- *(technical assumptions).*

Then $\widetilde{H}$ *is type-definable.*

Proof.

Using the technical theorem and a certain topology (weaker than Vietoris) on subsets of $A/(A \cap \widetilde{H})$. □

Main theorem for definable group extensions

Theorem

Suppose we have:

- $\widetilde{H} \trianglelefteq \widetilde{G}$: F_σ *normal subgroup;*
- $\widetilde{H} \cap A$ *and* $\pi[\widetilde{H}]$: *type-definable;*
- *(technical assumptions).*

Then $\widetilde{H}$ is type-definable.

Proof.

Using the technical theorem and a certain topology (weaker than Vietoris) on subsets of $A/(A \cap \widetilde{H})$. □

Main theorem for definable group extensions

Theorem

Suppose we have:

- $\widetilde{H} \trianglelefteq \widetilde{G}$: F_σ *normal subgroup;*
- $\widetilde{H} \cap A$ *and* $\pi[\widetilde{H}]$: *type-definable;*
- *(technical assumptions).*

Then $\widetilde{H}$ *is type-definable.*

Proof.

Using the technical theorem and a certain topology (weaker than Vietoris) on subsets of $A/(A \cap \widetilde{H})$. □

Main theorem for definable group extensions

Theorem

Suppose we have:

- $\widetilde{H} \trianglelefteq \widetilde{G}$: F_σ *normal subgroup;*
- $\widetilde{H} \cap A$ *and* $\pi[\widetilde{H}]$: *type-definable;*
- *(technical assumptions).*

Then $\widetilde{H}$ *is type-definable.*

Proof.

Using the technical theorem and a certain topology (weaker than Vietoris) on subsets of $A/(A \cap \widetilde{H})$. □

Corollary

Question (Gismatullin & Krupiński 2012)

In the meta-example, if $G^{00} = G^{000}$, does $\widetilde{G}^{000} \neq \widetilde{G}^{00}$ imply assumption † ?

Corollary (with some natural assumptions)

Yes!

Moreover, if $G^{00} = G^{000}$, then

$$\widetilde{G}^{00} / \widetilde{G}^{000} \cong K / D$$

where K is a compact group and D is finitely generated dense.

Corollary

Question (Gismatullin & Krupiński 2012)

In the meta-example, if $G^{00} = G^{000}$, does $\widetilde{G}^{000} \neq \widetilde{G}^{00}$ imply assumption $\dagger$?

Corollary (with some natural assumptions)

Yes!
Moreover, if $G^{00} = G^{000}$, then

$$\widetilde{G}^{00}/\widetilde{G}^{000} \cong K/D$$

where K is a compact group and D is finitely generated dense.

Infinite diameter is independent of n.f. (usually)

Example

- $T = \mathrm{Th}(\mathbf{R}, +, \cdot, 0, 1)$,
- E – total relation,
- $\Phi_n(x, y) = \bigvee_n \bigwedge_{m \geq n} (x = m \leftrightarrow y = m)$ is a normal form for E,
- E has only one class, which has infinite diameter w.r.t. $\bigvee_n \Phi_n$;
- the only class clearly has diameter 1 with respect to trivial normal form $\Phi'_n(x, y) = \top$
- E does not refine $\equiv$.

Example

$$G = \langle (1,2)(3,5)(4,6), (1,3,6)(2,4,5) \rangle$$
$$= \{(), (1,2)(3,5)(4,6), (1,3,6)(2,4,5),$$
$$(1,4)(2,3)(5,6), (1,5)(2,6)(3,4), (1,6,3)(2,5,4)\}$$

- $1 \sim 2$, $3 \sim 4$, $5 \sim 6$ (and no other nontrivial relations)
- M: (finite) structure such that G is the automorphism group and $\sim$ is definable.