# Topological dynamics and the complexity of strong types II

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# Motivations

- Characterising type-definability (and relative definability) of invariant equivalence relations, in countable and uncountable case.
- In particular, generalising the following fact, as well as previous results of [Krupiński–Rz.] and [Kaplan–Miller].

**Fact (Newelski)**

*If $E$ is an $F_\sigma$ equivalence relation on a set $X = p(\mathfrak{C})$ for some $p \in S(\emptyset)$, while $Y \subseteq X$ is type-definable and $E$-saturated, then if $|Y/E| < 2^{\aleph_0}$, then $E$ is type-definable.*

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# General ideas

- Idea: use facts about compact groups to deduce facts about bounded invariant equivalence relations.
- Problem: Galois groups and type spaces are not Hausdorff, in general.
- How to avoid the problem? Using topological dynamics.

$$\begin{array}{ccccc} (u\mathcal{M}) & \xrightarrow{j} & (u\mathcal{M})/H(u\mathcal{M}) & \xrightarrow{\bar{f}} & \mathrm{Gal}_L(T) \\ \downarrow r & \searrow^{h_E} & & \searrow^{\bar{h}_E} & \downarrow g_E \\ X_M & & \longrightarrow & & X/E \end{array}$$

# Mycielski's theorem

## Proposition

*Suppose $E$ is a meagre equivalence relation on a compact space $X$. Then $E$ has at least $2^{\aleph_0}$-many classes.*

## Proof.

- Suppose $E \subseteq \bigcup_{n\in\mathbf{N}} F_n \subseteq X^2$, where $F_n$ are closed nowhere dense and non-decreasing
- We define recursively a family $U_s$, $s \in 2^{<\omega}$, so that
  1. $\forall s$, $\overline{U_{s0}}, \overline{U_{s1}} \subseteq U_s$,
  2. if $s \neq t$ and $s, t \in 2^{n+1}$, then $(U_s \times U_t) \cap F_{n+1} = \emptyset$.
- The construction is straightforward.
- Picking for each $\eta \in 2^\omega$ an arbitrary point in $\bigcap_{n\in\omega} U_{\eta\restriction n}$ (which exists by compactness) completes the proof. □

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# The case of compact groups

## Corollary

*Suppose $H$ is a meagre subgroup of a compact Hausdorff group $G$. Then $[G : H]$ is at least $2^{\aleph_0}$.*

## Proof.

- Notice that the map $(x, y) \mapsto xy^{-1}$ is continuous and open, so preimages of meagre sets are meagre.
- In particular, the relation of lying in the same coset of $H$ is meagre, and we can apply the proposition. □

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# The key corollary

**Fact (Piccard-Pettis theorem)**

*If $A \subseteq G$ is a nonmeagre and Baire (i.e. closed modulo meagre) subset of a [semi]topological group, then $AA^{-1}$ contains a neighbourhood of $e$.*

**Corollary (Key corollary)**

*If $G$ is a compact Hausdorff group and $H \leq G$ is Baire and not open, then $[G : H]$ is at least $2^{\aleph_0}$ (if $H$ is open, $[G : H]$ is finite).*

**Proof.**

By the fact, if $H$ is not open, it must be meagre. Then the preceding corollary applies immediately. □

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# Souslin operation

## Definition

Suppose $(A_s)_{s\in\omega^{<\omega}}$ is a tree of subsets of a set $X$. Then we define the Souslin operation by

$$\mathcal{A}_s A_s = \bigcup_{\eta\in\omega^\omega} \bigcap_{n\in\omega} A_{\eta\restriction n}$$

If $A_s$ are in a fixed class $\mathcal{C}$ of subsets of $X$, we say that $\mathcal{A}_s A_s$ is Souslin over $\mathcal{C}$.

## Fact

*The Souslin operation applied to a family of [strictly] Baire subsets of a topological space (i.e. closed modulo meagre [in every subspace]) is [strictly] Baire (i.e. all sets Souslin over [strictly] Baire sets are themselves [strictly] Baire).*

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# Uncountable language case

## Theorem

*We are working in the monster model $\mathfrak{C}$ of a complete theory. Let $p \in S(\emptyset)$. Suppose we have:*

- *a bounded, invariant equivalence relation $E$ on $X = p(\mathfrak{C})$, which is Souslin over type-definable sets (e.g. $E$ is $F_\sigma$),*
- *a type-definable and $E$-saturated set $Y \subseteq X$.*

*Then:*

(I) *$E$ is type-definable, or $E{\restriction}_Y$ has at least $2^{\aleph_0}$-many classes,*

(II) *in addition, if $\mathrm{Aut}(\mathfrak{C}/\{Y\})$ acts transitively on $Y/E$ (e.g. $Y = p(\mathfrak{C})$ or $Y$ is a KP strong type), then either $E{\restriction}_Y$ is (relatively) definable (so, by compactness, it has finitely many classes), or $E{\restriction}_Y$ has at least $2^{\aleph_0}$-many classes.*

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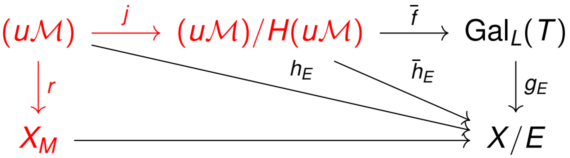
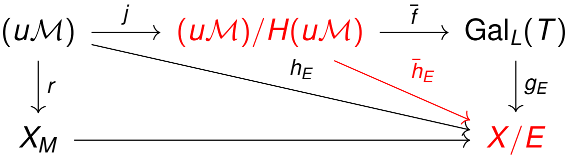
# Strict Baire property




## Lemma

*If $E$ is as in the theorem (i.e. Souslin over type-definable sets), then for any fixed $\bar{\alpha} \in X$, the $E$-class of $\bar{\alpha}$ is Souslin over type-definable sets, while the “kernel” of $\bar{h}_E$ is Souslin over closed sets, and in particular strictly Baire.*

# (Not) openness in case of $Y = X\ [= p(\mathfrak{C})]$




## Theorem (QM theorem)

*$\bar{h}_E$ is a topological group quotient mapping.*

## Corollary

*If $|X/E| < 2^{\aleph_0}$, then (by QM) also $[u\mathcal{M}/H(u\mathcal{M}) : \ker \bar{h}_E] < 2^{\aleph_0}$, so $\ker \bar{h}_E$ is open. This implies (by QM) that $X/E$ is discrete (and compact), so $E$ is relatively definable.*

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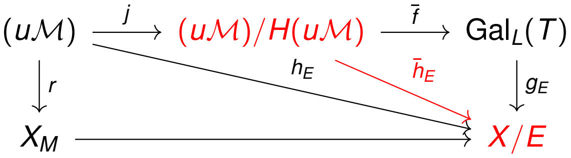


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# The idea for the case of $Y \subsetneq X\,[= p(\mathfrak{C})]$, part I

$$\ker \bar{h}_E \leq (u\mathcal{M})/H(u\mathcal{M}) \xrightarrow{\bar{h}_E} X/E$$

$$\ker \bar{h}_E \leq G_1 \xrightarrow{\bar{h}_E\restriction_{G_1}} Y/E$$

- Let $G_1$ be the closure of $\ker \bar{h}_E$, a closed subgroup of $(u\mathcal{M})/H(u\mathcal{M})$.
- Clearly $G_1 \subseteq \bar{h}_E^{-1}[Y/E]$ (by continuity of $\bar{h}_E$).
- If $E$ is not type-definable, $X/E$ is not Hausdorff, so by the QM theorem, $\ker \bar{h}_E$ is not closed in $(u\mathcal{M})/H(u\mathcal{M})$ (so also not closed in $G_1$).
- Then, by the key corollary, $[G_1 : \ker \bar{h}_E] \geq 2^{\aleph_0}$.
- This clearly implies that $|Y/E| \geq 2^{\aleph_0}$.

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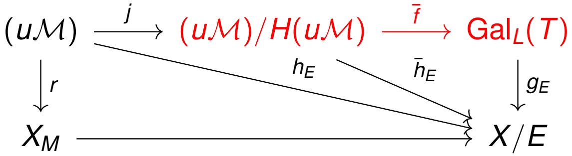
# Subgroups of $(u\mathcal{M})/H(u\mathcal{M})$




- $\mathrm{Aut}(\mathfrak{C})$ acts on $X = p(\mathfrak{C})$, which induces an action of $\mathrm{Gal}_L(T)$ on $X/E$.
- For a type-definable and $E$-saturated $Z \subseteq X$, the stabiliser in $\mathrm{Gal}_L(T)$ of $\{Z/E\}$ is closed.
- The preimage by $\bar{f}$ of a closed subgroup of $\mathrm{Gal}_L(T)$ is a closed subgroup of $(u\mathcal{M})/H(u\mathcal{M})$.

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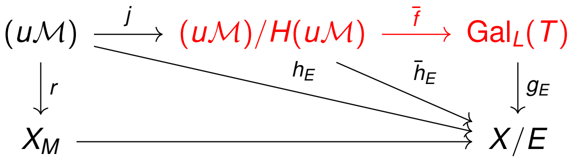


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# The idea for the case of $Y \subsetneq X\, [= p(\mathfrak{C})]$, part II

$$\ker \bar{h}_E \leq (u\mathcal{M})/H(u\mathcal{M}) \xrightarrow{\bar{h}_E} X/E$$

$$\ker \bar{h}_E \cap G_2 = \ker(\bar{h}_E\restriction_{G_2}) \leq G_2 \xrightarrow{\bar{h}_E\restriction_{G_2}} Y/E$$

- We relativise to the (closed) subgroup $G_2$ of $(u\mathcal{M})/H(u\mathcal{M})$ induced by $\mathrm{Aut}(\mathfrak{C}/\{Y\})$ (i.e. the $\bar{f}$-preimage of the stabiliser of $\{Y/E\}$ in $\mathrm{Gal}_L(T)$).
- By the assumption, $\bar{h}_E\restriction_{G_2}$ is onto $Y/E$.
- By part (I), if $|Y/E| < 2^{\aleph_0}$, $X/E$ is Hausdorff, so $\bar{h}_E\restriction_{G_2}$ is a quotient mapping onto $Y/E$ (as a continuous surjection).
- Then, by the key corollary, if $|Y/E| < 2^{\aleph_0}$, then $\ker\left(\bar{h}_E\restriction_{G_2}\right)$ is open, so $Y/E$ is discrete and $E\restriction_Y$ is relatively definable.

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- We relativise to the (closed) subgroup $G_2$ of $(u\mathcal{M})/H(u\mathcal{M})$ induced by $\mathrm{Aut}(\mathfrak{C}/\{Y\})$ (i.e. the $\bar{f}$-preimage of the stabiliser of $\{Y/E\}$ in $\mathrm{Gal}_L(T)$).
- By the assumption, $\bar{h}_E{\restriction}_{G_2}$ is onto $Y/E$.
- By part (I), if $|Y/E| < 2^{\aleph_0}$, $X/E$ is Hausdorff, so $\bar{h}_E{\restriction}_{G_2}$ is a quotient mapping onto $Y/E$ (as a continuous surjection).
- Then, by the key corollary, if $|Y/E| < 2^{\aleph_0}$, then $\ker\left(\bar{h}_E{\restriction}_{G_2}\right)$ is open, so $Y/E$ is discrete and $E{\restriction}_Y$ is relatively definable.

# The idea for the case of $Y \subsetneq X\, [= p(\mathfrak{C})]$, part II

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# The main general theorem – reminder

**Theorem**

*We are working in the monster model $\mathfrak{C}$ of a complete, countable theory. Let $p \in S(\emptyset)$. Suppose we have:*

- *a bounded, invariant equivalence relation $E$ on $p(\mathfrak{C})$,*
- *a type-definable and $E$-saturated set $Y \subseteq p(\mathfrak{C})$.*

*Then, $E{\restriction}_Y$ is either type-definable or non-smooth.*

# The trichotomy theorem

## Corollary

*Assume that the language is countable. Let $E$ be a bounded, Borel (or even analytic) equivalence relation on $p(\mathfrak{C})$, where $p \in S(\emptyset)$. Then, exactly one of the following holds:*

1. *$E$ is relatively definable (on $p(\mathfrak{C})$), smooth, and has finitely many classes,*
2. *$E$ is not relatively definable, but it is type-definable, smooth, and has $2^{\aleph_0}$ classes,*
3. *$E$ is not type definable, non-smooth, and has $2^{\aleph_0}$ classes.*

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# Proof of the trichotomy theorem

- If $E$ has less than continuum many classes, then by the preceding theorem, it must be relatively definable (and thus it has finitely many classes, by compactness).
- Otherwise, $E$ must have $2^{\aleph_0}$ classes (as it can't have any more by countability assumptions).
- By the main theorem about smoothness of Borel equivalence relations, $E$ is smooth if and only if it is type-definable.

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# Neccessity of the assumptions – regularity

**Example (Kaplan–Miller–Simon)**

There is a definable group $G$ in a countable theory with an invariant subgroup $H \leq G$ of index 2 which is not type-definable.

**Corollary**

*The discussed theorems do not hold in general without any regularity (e.g. Borelness, analyticity) assumptions about $E$.*

**Proof.**

If we add a sort for an "affine copy of $G$", the resulting structure will have an invariant equivalence relation with two classes (corresponding to $H$), whose domain is the set of the realisations of a single type, but which is not type-definable. □

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# Neccessity of the assumptions – transitive action of $\mathrm{Aut}(\mathfrak{C}/\{Y\})$

**Example**

- $T = \mathrm{Th}(2^\omega, E_n)_{n\in\omega}$, where $E_n$ is the equality on the $n$-th coordinate,
- $E = \bigcap_n E_n$.
- Then $\mathfrak{C}/E \approx 2^\omega$.
- Let $Y \subseteq \mathfrak{C}$ correspond to a convergent sequence along with its limit.
- Then $Y$ is type-definable and $|Y/E| = \aleph_0$, so $E{\restriction}_Y$ is not relatively definable.

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