Uniform distribution of the Fibbonacci sequence

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Note: **In this paper we use a position system with a base five.**

Theorem. *For each* $j = 0, 1, 2, 3$ *the sequence* F_{4n+j} *is uniformelly distributed modulo* 10^N. *Moreover it is a permutation sequence.*

Corrolary. *The Fibbonacci sequence is uniformelly distributed modulo* 10^N.

Proof of the Theorem:

Claim 1. $F_{k+13} = 12F_{k+4} - F_k$.

This follows by induction #.

Define $\mathcal{G}_4 := \begin{pmatrix} 12 & -1 \\ 1 & 0 \end{pmatrix}$. Then $\mathcal{G}_4 \begin{pmatrix} F_{k+4} \\ F_k \end{pmatrix} = \begin{pmatrix} F_{k+13} \\ F_{k+4} \end{pmatrix}$.

Claim 2. $\mathcal{G}_4^{10^N} \equiv 1 + 10^N(\mathcal{G}_4 - 1)(\mathrm{mod} 10^{N+1})$.

This follows by induction. Define $R_N = \left(\mathcal{G}_4^{10^N} - 1 - 10^N(\mathcal{G}_4 - 1)\right)/10^{N+1}$. One needs to rize $\mathcal{G}_4^{10^N} = 1 + 10^N(\mathcal{G}_4 - 1) + 10^{N+1}R_N$ to the 10th power and examine Newton coefficients using the fact that $10|(\mathcal{G}_4 - 1)^2$ #.

As a corrolary we see $F_{j+10^N} \equiv F_j + 10^N(F_{j+4} - F_j)(\mathrm{mod} 10^{N+1})$, and what follows $F_{j+k10^N} \equiv F_j + k10^N(F_{j+4} - F_j)(\mathrm{mod} 10^{N+1})$. By hypothesis $F_{4n+j} \equiv F_{4n+j+k10^N}(\mathrm{mod} 10^N)$ gives all residue classes modulo 10^N for $n = 1, \ldots, 10^N$. But $F_{4n+j+k10^N}$ are all different $\mathrm{mod} 10^{N+1}$ for $k = 1, \ldots, 4$, since $10 \not| F_{j+4} - F_j$. This proves the theorem.

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