Counting faces of flag spheres

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or at arXiv:math.CO/0501046

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Let L be a finite simplicial complex with a vertex set S.

Definition. *f-polynomial of a simplicial complex L is a generating function defined as:*

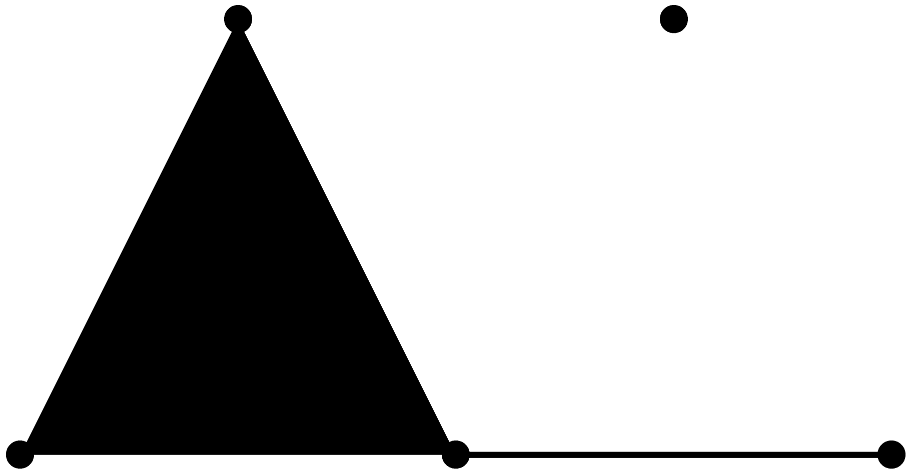
$$f_L(t) := \sum_{\sigma \in L} t^{\#\sigma} = \sum_i f_i t^i,$$

$$f_i := \#\{\text{simplices with } i \text{ vertices}\}.$$

Questions:

- What can be said in general about f-polynomials of simplicial complexes?
- What can be said in general about f-polynomials of certain classes of simplicial complexes (eg. triangulations of spheres or flag triangulations of spheres)?

Examples:

$L =$ 

- $f_L = 1 + 5t + 4t^2 + t^3$.
- dodecahedron: $f_{I_{12}} = 1 + 20t + 30t^2 + 12t^3$.
- boundary of k-simplex: $f_{\partial\Delta^k} = (1+t)^{k+1} - t^k$,
- boundary of a cross-poytope: $f_{O^k} = (1+2t)^k$.

Definition. *A simplicial complex L is called flag if for any clique T (a subset $T \subset S$ such that any two vertices of T are joined by an edge) T is a face of L.*

Example. *The baricentric subdivision of a regular CW-complex (eg. polytopial complex) is a flag simplicial complex.*

The construction of M. Davis associates with any flag triangulation of a sphere L a compact aspherical manifold M_L. There is defined a cubing of M_L (M_L is a cubical complex), such that the link of any vertex in M_L is isomorphic to L.

Conjecture (Hopf). *The Euler Characteristic of a compact aspherical manifold M of dimension $2n$ fulfils*

$$(-1)^n \chi(M) \geq 0.$$

In the case of M_L the above Corollary admits the following reformulation:

Conjecture (Charney-Davis). *Assume that L is a flag triangulation of S^{2n-1}. Then*

$$(-1)^n f_L\left(-\frac{1}{2}\right) \geq 0.$$

The Charney-Davis Conjecture is known to be true for

- $n = 1$,
- $n = 2$ (Davis i Okun, 2001),
- barycentric subdivisions (Babson, Stanley 1994, Okun 2001, Karu 2004),
- locally convex triangulations of spheres (Reiner-Leung 2002).

Useful change of variables

Definition. *h-polynomial of an $(n-1)$-dimensional complex L is given by the formula*

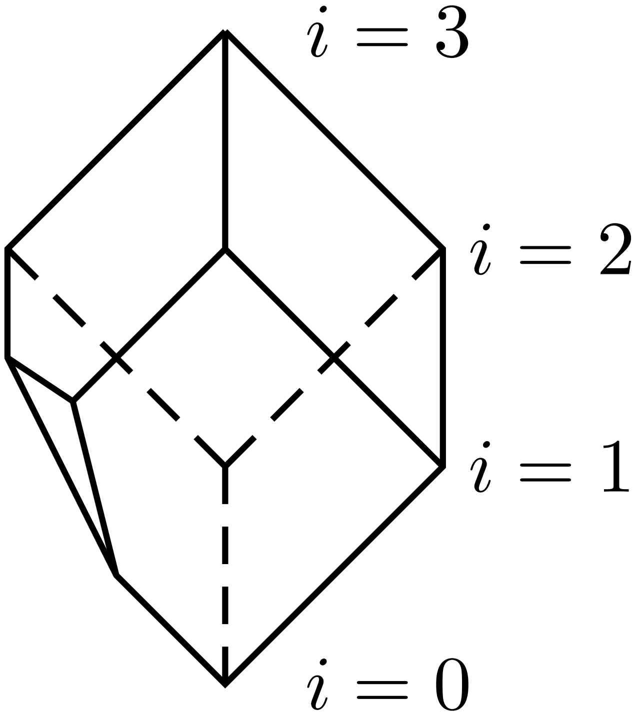
$$(1+t)^n h_L\left(\frac{1}{1+t}\right) = t^n f_L(1/t).$$

h-polynomial is more efficient:

- if L is a triangulation of S^{n-1} then h_L is reciprocal (palindromic), i.e. $h_L(1/t) = t^n h_L(t)$),
- if L is a convex triangulation of a sphere, then h_L is unimodal (e.g $h_{\lceil \frac{n}{2} \rceil} \geq \dots h_2 \geq h_1 \geq h_0 = 1$),
- the Charney-Davis Conjecture holds for L $\iff$

$$(-1)^n h_L(-1) \geq 0,$$

- when L is a convex triangulation of a sphere then h_L has a simple interpretation in terms of the height function.



$$h = 1 + 4\,t + 4\,t^2 + t^3$$

- if L is a triangulation of S^{n-1} with m vertices, then

$$h_i \leq \binom{m-n+i-1}{i}$$

for $2i < n$,

Real Root Conjecture (2003). *If L is a flag triangulation of a sphere/GHS then h_L (f_L) has only real non-positive roots.*

The Real Root Conjecture would imply

- unimodularity of h-polynomials,
- Charney-Davis Conjecture,
- Neggers-Stanley Conjecture for naturally labeled posets of width 2.

Fact. *When L is a flag complex (different from a simplex) then h_L has a real root.*

Fact. *When L is a flag complex (different from the join of a simplex and a cross-polytope) then h_L has a real root different from -1.*

Corollary. *The Real Root Conjecture holds for triangulations of S^1 and S^2.*

Corollary (from the Davis-Okun Theorem). *The Root Conjecture holds for triangulations of* S^3

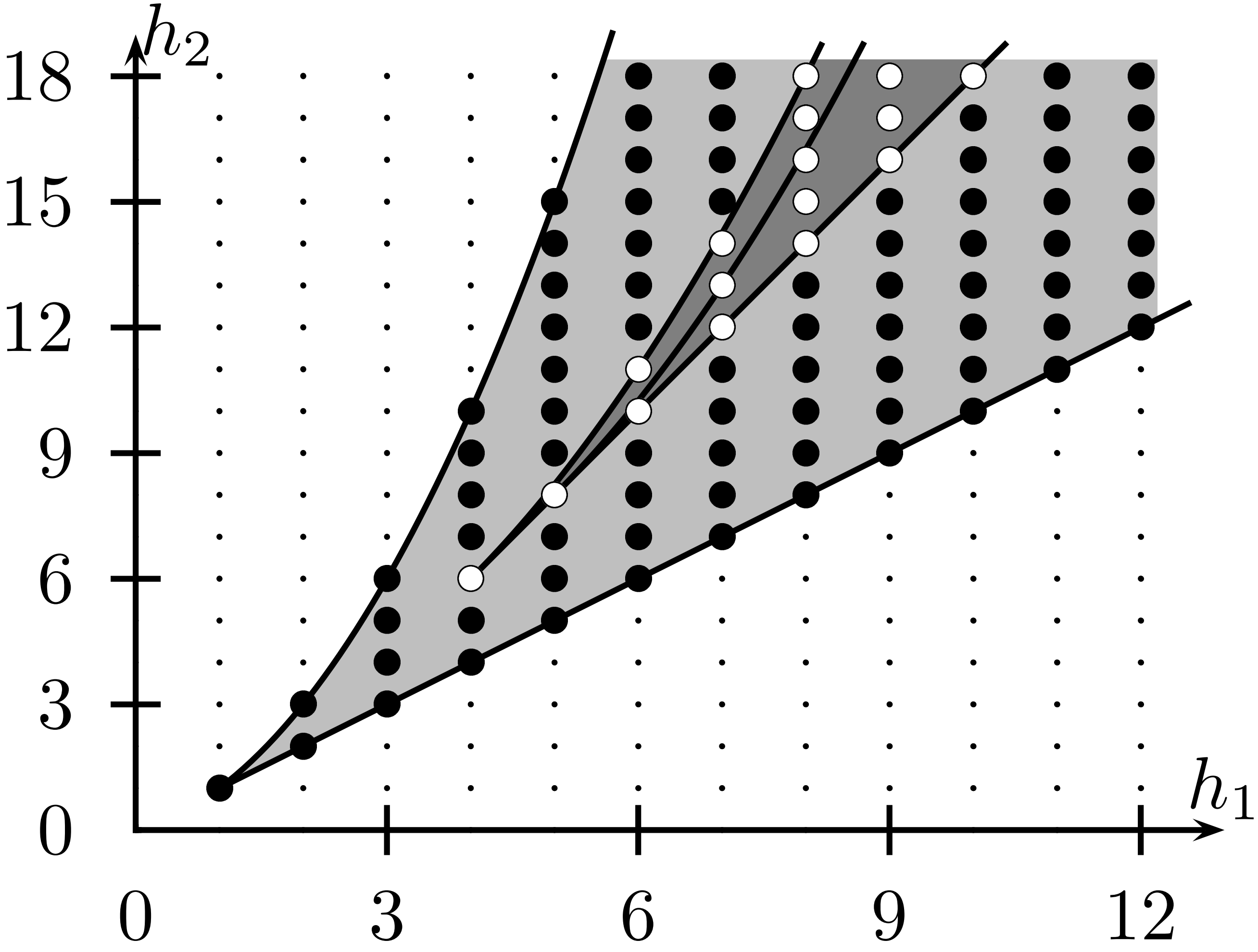
Theorem (G.). *The Root Conjecture holds for triangulations of* S^4

Proof: It is easy to find three real roots.

The sum of f-polynomials of links of all vertices of L equals to f_L'.

The links of L fulfils the assumptions of the Davis-Okun Theorem, thus computing the sign of $f_L'(-1/2)$ we obtain the claim.

Theorem (G.). *Let h be a quartic monic reciprocal polynomial with integer polynomials. If $h(t) - t(1+t)^2$ has only real non-positive roots then h is a h-polynomial of a (convex) triangulation of S^3.*

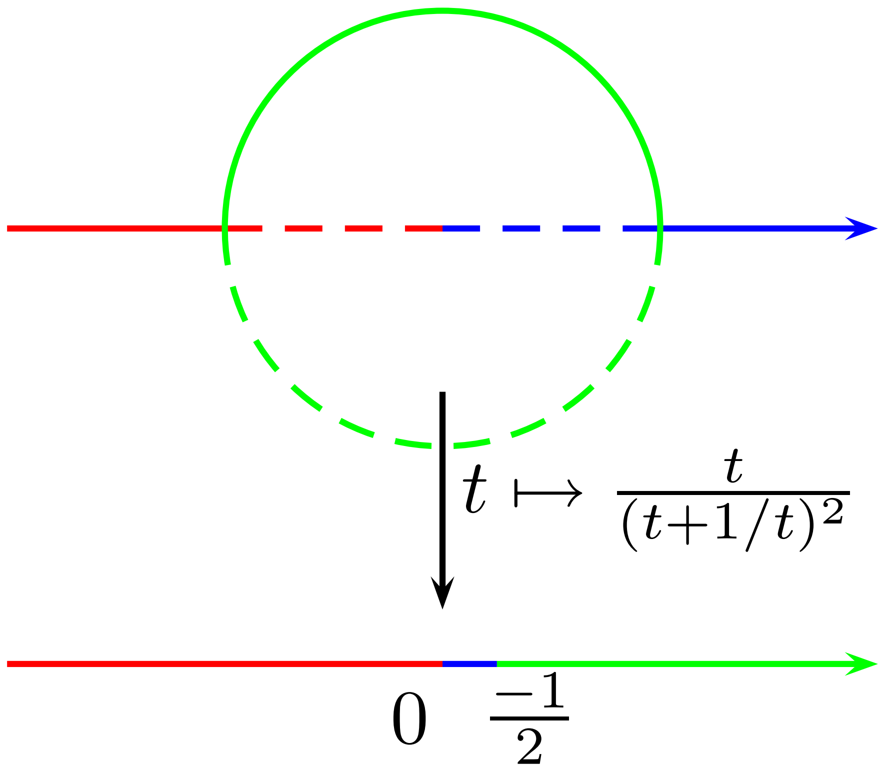


Open Question. *Is* $1 + 13\,t + 54\,t^2 + 82\,t^3 + 41\,t^4$ *(the mean of f-polynomials of joins of five- and eightgons and six- and sevengons) an f-polynomial of a flag triangulation of a sphere?*

Reciprocal polynomial h may be expressed as

$$h(t) = (1+t)^n \gamma\left(\frac{t}{(t+t^{-1})^2}\right),$$

where γ has degree $\lfloor \deg h/2 \rfloor$.



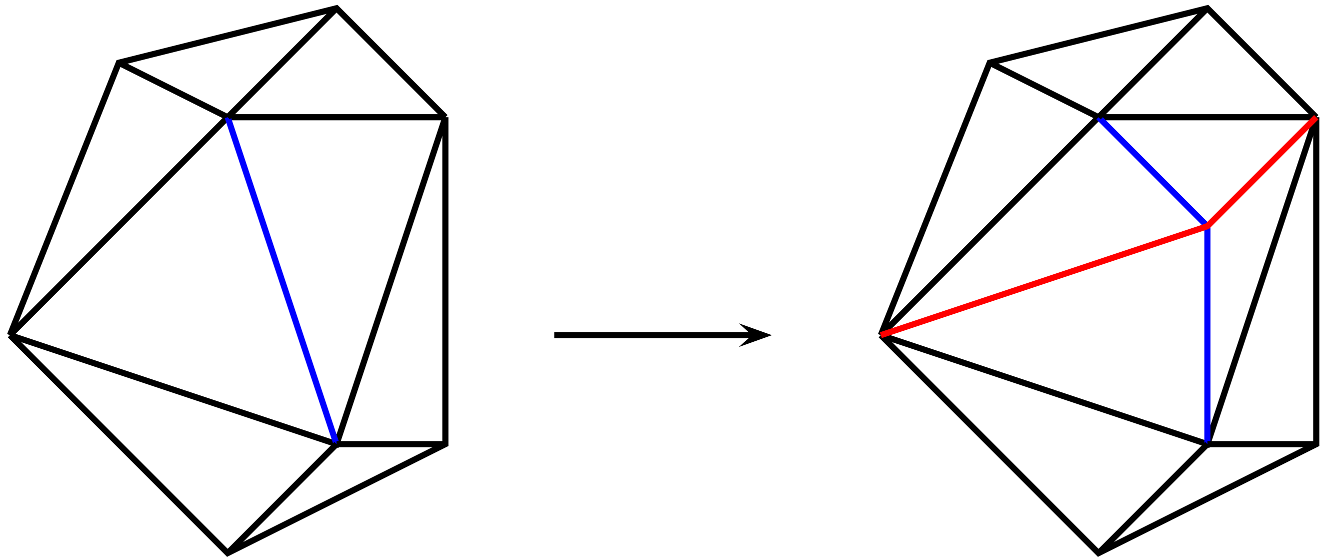
What is known about γ_L?

- γ_L has only real nonpositive roots if and only if h_L does,
- Charney-Davis Conjecture is equivalent to the non-negativity of the highest coefficient of γ_L,
- $\gamma_1(L) \geq 0$,
- γ_L has at least one nonpositive real root unless $\gamma_1(L) = 0$ ($\iff \gamma_L \equiv 1 \iff L$ is a cross-polytope),
- if L is flag triangulation of S^{n-1} and $n \leq 5$, then γ_L has nonnegative coefficients,
- if L is a baricentric subdivision of a convex polytope (regular CW complex) P then γ_L has all coefficients nonnegative, moreover if Φ_P is a cd-index of P then

$$\gamma_L(t) = \Phi_P(1, 2t).$$

Fact. *$f_\bullet$, $h_\bullet$, $\gamma_\bullet$ are multiplicative with respect to the operation of join.*

Fact. *If $\mathrm{Sub}_e\, L$ is a subdivision of L along an edge e. Then* $\gamma_{\mathrm{Sub}_e\, L}(t) = \gamma_L(t) + t\gamma_{Lk(e)}(t)$.



Corollary. *If γ-polynomials of both L and $\mathrm{Lk}(e)$ have non-negative coefficients then so does also $\gamma_{\mathrm{Sub}_e\, L}$.*

Conjecture (G.). *All coefficients of* γ_L *are nonnegative when* L *is flag triangulation of a sphere/GHS.*

Question. *What is the combinatorial/geometric interpretation of the coefficients of* γ*-polynomial?*

Fact. *If the polynomial* $1 + g_1 t + g_2 t^2 + g_3 t^3 = (1 + xt)(1 + yt)(1 + zt)$ *has only real roots, then*

$$g_2^2 \geq 3 g_3 g_1.$$

Proof :

$$2\left((xy + yz + zx)^2 - 3xyz(x + y + z)\right) = x^2(y - z)^2 + y^2(z - x)^2 + z^2(x - y)^2.$$

Corollary (G). *If L is a flag triangulation of S^5, such that, $h_L(-1) < 0$ and some edge has a link, which is a crossed polytope, then a (multiple) subdivision along this edge provides a counterexample to the Real Root Conjecture.*

Recollection : $\gamma_{\mathrm{Sub}_e\, L}(t) = \gamma_L(t) + t\gamma_{\mathrm{Lk}(e)}(t)$.

Let V be a 5-gon. Let W be a subdivision of $V * V$ along an edge, whose link is a quadrilateral. Then $L = V * W$ fulfils the assumptions of the above Corollary.

Remark. *The operation of a subdivision along an edge may be done geometrically. That means, if L can be realized as a boundary of a convex polytope, ten so does $\mathrm{Sub}_e\, L$.*